Specialization maps for Scholze's category of diamonds.

By

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Abstract

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The purpose of this thesis is to introduce and study the specialization map in the context of Scholze's category of diamonds and to prove some basic results on its behavior. Our specialization map generalizes the classical specialization map that appears in the theory of formal schemes. Afterwards, as an example of interest, we study the specialization map for p-adic Beilinson-Drinfeld Grassmanians and moduli spaces of mixed-characteristic shtukas associated to reductive groups over $\mathbb{Z}_p$. Finally, as an application of our theory, we describe the geometric connected components of some moduli spaces of mixed-characteristic shtukas and local Shimura varieties at infinite level. This confirms and generalizes conjecture 4.26 of [46] in the unramified case.

Dedicado con amor a mis padres Laura y Eduardo.

Contents

1 Theoretical aspects of the specialization map **2**
- 1.1 The v-topology . . . 2
  - 1.1.1 Diamonds and small v-sheaves . . . 2
  - 1.1.2 Spectral spaces and locally spatial diamonds . . . 7
  - 1.1.3 Pre-adic spaces as v-sheaves . . . 9
- 1.2 The olivine spectrum . . . 11
  - 1.2.1 Review, terminology and conventions . . . 12
  - 1.2.2 Definitions and basic properties . . . 13
  - 1.2.3 Olivine Huber pairs . . . 18
  - 1.2.4 Some open and closed subsheaves of $\mathrm{Spd}(B, B^+)$ . . . 25
  - 1.2.5 Discrete Huber pairs in characteristic p . . . 28
- 1.3 The reduction functor . . . 35
  - 1.3.1 The v-topology for perfect schemes . . . 35
  - 1.3.2 Reduction functor and formal adicness . . . 43
- 1.4 Specialization . . . 49
  - 1.4.1 Specialization for Tate Huber pairs . . . 50
  - 1.4.2 Specializing v-sheaves . . . 51
  - 1.4.3 Kimberlites, formal schemes and tubular neighborhoods . . . 55
  - 1.4.4 cJ-diamonds and rich kimberlites . . . 61

2 Specialization maps for some moduli problems **67**
- 2.1 G-torsors, lattices and shtukas . . . 67
  - 2.1.1 Vector bundles, torsors and meromorphicity . . . 67
  - 2.1.2 Vector bundles on $\mathcal{Y}$ . . . 69
  - 2.1.3 Lattices and shtukas . . . 74
- 2.2 The specialization map for p-adic Beilinson-Drinfeld Grassmanians . . . 77
  - 2.2.1 Grassmanians as kimberlites . . . 77
  - 2.2.2 Twisted loop sheaves . . . 81
  - 2.2.3 Demazure kimberlites . . . 86
  - 2.2.4 Resolution of p-adic Beilinson-Drinfeld Grassmanians . . . 97
- 2.3 Specialization for moduli of mixed characteristic shtukas . . . 101
  - 2.3.1 Moduli spaces of shtukas are kimberlites . . . 102
  - 2.3.2 Comparison of tubular neighborhoods . . . 109

3 Geometric connected components at infinite level. 117
3.1 Notation . . . 117
3.2 The geometric perspective of crystalline representations . . . 118
3.2.1 Vector bundles, isocrystals and crystalline representations. . . . 118
3.2.2 Families of B_{dR}-lattices . . . 120
3.2.3 Isocrystals with G-structure. . . . 123
3.2.4 G-bundles and G-valued crystalline representations . . . 125
3.2.5 M. Chen's result on p-adic Hodge Theory . . . 131
3.2.6 The geometric realization of $\mathbb{L}$ and p-adic shtukas . . . 132
3.2.7 Weil descent . . . 133
3.2.8 The action of $J_b(\mathbb{Q}_p)$ . . . 137
3.2.9 Group functoriality . . . 141
3.3 The case of tori . . . 142
3.3.1 Norm morphisms . . . 142
3.3.2 The Weil group action on the Lubin-Tate case . . . 146
3.4 On the unramified case. . . . 149
3.4.1 Connected components of affine Deligne Lusztig Varieties . . . 150
3.4.2 The simply connected case . . . 152
3.4.3 z-extensions . . . 157

Introduction

The purpose of this thesis is threefold, and each goal corresponds to a chapter. In the first chapter we construct the specialization map in the context of Scholze's category of diamonds, and we study abstract properties and related constructions in a very theoretical framework. In the second chapter, we apply the theory developed in the first chapter to study the specialization map for the p-adic Beilinson-Drinfeld Grassmanians and moduli spaces of mixed-characteristic (or p-adic) shtukas. These moduli spaces were introduced in the Berkeley notes ([53]) as some of the most important examples that motivated the development of the theory of diamonds. In the final chapter we use our finding from the second chapter to explicitly describe the structure of the set of connected components of a big class of local Shimura varieties and moduli spaces of mixed-characteristic shtukas at infinite level.

To fix ideas let us recall the specialization map in a more classical setup. Let $\mathcal{X}$ be a separated formal scheme topologically of finite type over $\mathbb{Z}_p$. One can associate to $\mathcal{X}$ a rigid analytic space over $\mathbb{Q}_p$, that we will denote by $X_\eta := \mathcal{X} \times_{\mathbb{Z}_p} \mathbb{Q}_p$. We can also associate to $\mathcal{X}$ a finite type reduced scheme over $\mathbb{F}_p$, that we denote by $\overline{X} := (\mathcal{X} \times_{\mathbb{Z}_p} \mathbb{F}_p)^{\mathrm{red}}$. Now, Huber's theory of adic spaces allows us to consider X_η as an adic space and in particular assign to it a locally spectral topological space $|X_\eta|$. Moreover, one can construct a continuous and spectral map of locally spectral spaces $\mathrm{sp}_{\mathcal{X}} : |X_\eta| \to |\overline{X}|$, where $|\overline{X}|$ is the usual Zariski space underlying $\overline{X}$ (See [4] 7.4.12 or [36] 6.4). It is this specialization map that our work generalizes, we elaborate below.

In [51] Scholze sets foundations for the theory of diamonds which can be defined as certain sheaves on the category of characteristic p perfectoid spaces endowed with a Grothendieck topology called the v-topology. He associates to any pre-adic space X over $\mathbb{Z}_p$ (not necessarily analytic) a v-sheaf $X^\diamondsuit$, and whenever X is analytic he proves that $X^\diamondsuit$ is a (locally spatial) diamond. Moreover, Scholze assigns to any v-sheaf $\mathcal{F}$ an underlying topological space $|\mathcal{F}|$ and whenever $\mathcal{F} = X^\diamondsuit$ he constructs a functorial surjective and continuous map $|\mathcal{F}| \to |X|$. When X is analytic it is proven in [53] that this map is a homeomorphism, but as we will discuss below this fails for non-analytic pre-adic spaces.

In the first chapter, we take as input what we call below a *specializing* v-sheaf $\mathcal{F}$ and we assign to it: a scheme-theoretic v-sheaf $\mathcal{F}^{\mathrm{red}}$ which is the analogue of the reduced special fiber of a formal scheme, and a continuous map of topological spaces $\mathrm{sp}_\mathcal{F} : |\mathcal{F}| \to |\mathcal{F}^{\mathrm{red}}|$ that we call the *specialization map* of $\mathcal{F}$. If $\mathcal{X}$ is a separated formal scheme over $\mathbb{Z}_p$, we can prove that $\mathcal{X}^\diamondsuit$ is a specializing v-sheaf, and in this case we have natural identifications $|\mathcal{X}_\eta| = |\mathcal{X}^\diamondsuit \times_{\mathbb{Z}_p^\diamondsuit} \mathbb{Q}_p^\diamondsuit|$ and $|\overline{X}| = |(\mathcal{X}^\diamondsuit)^{\mathrm{red}}|$ together with a commutative diagram:

$$\begin{array}{ccc} |\ \mathcal{X}^\diamondsuit \times_{\mathbb{Z}_p^\diamondsuit} \mathbb{Q}_p^\diamondsuit\ | & \xrightarrow{\cong} & |\ \mathcal{X}_\eta\ | \\ \downarrow{\scriptstyle \mathrm{sp}_{\mathcal{X}^\diamondsuit}} & & \downarrow{\scriptstyle \mathrm{sp}_{\mathcal{X}}} \\ |\ (\mathcal{X}^\diamondsuit)^{\mathrm{red}}\ | & \xrightarrow{\cong} & |\ \overline{X}\ | \end{array}$$

It is in this sense that our specialization map generalizes the classical one.

The advantage of working in this broader context is that the categories of diamonds and v-sheaves are much more flexible than those of formal schemes and rigid analytic spaces. This allows us to construct interesting spaces that do not come from applying the $\diamondsuit$-functor to pre-adic spaces. Actually, the main reason the author found the specialization map for diamonds interesting is that it has applications to the study of moduli spaces of mixed-characteristic shtukas. Typically, these moduli spaces are locally spatial diamonds that do not come from a pre-adic space. In forthcoming work of the author, we use the tools developed here to describe the profinite set of geometric connected components of some moduli spaces of p-adic shtukas at any chosen level (including infinite level). This work builds on and generalizes the work of Chen on the geometric connected components of unramified Rapoport-Zink spaces (See [8]).

To describe the main results of our second chapter, we fix some notation. Let $\mathscr{G}$ be a reductive group over $\mathbb{Z}_p$, and denote by G the generic fiber of $\mathscr{G}$ over $\mathbb{Q}_p$. Fix $T \subseteq B \subseteq G$ a maximal $\mathbb{Q}_p$-rationally defined torus and a Borel respectively, and let $\mathfrak{f}$ be an algebraically closed field extension of $\overline{\mathbb{F}}_p$. We let X_*^+ denote the subset of dominant cocharacters in $X_*(T_{\overline{\mathbb{Q}}_p})$, fix a $\mu \in X_*^+$ and an element $b \in G(W(\mathfrak{f})[\frac{1}{p}])$. Let $E := E(\mu)$ be the reflex field of μ. Since $\mathscr{G}$ is reductive over $\mathbb{Z}_p$ this is an unramified extension of $\mathbb{Q}_p$. Let J_b denote the σ-centralizer of b appearing in Kottwitz' theory of isocrystals with G-structure [33]. Let F_1 denote a complete non-Archimedean field extension of E, with ring of integers of O_{F_1} and residue field k_{F_1}. Let F_2 be a complete non-Archimedean field extension of $W(\mathfrak{f})[\frac{1}{p}]$ with residue field k_{F_2}. To this data one can associate the following objects:

a.- A spatial diamond $\mathrm{Gr}^{G,\leq\mu}_{F_1^\diamond}$ proper over $F_1{}^\diamond$, parametrizing μ-bounded B^+_{dR}-lattices with G-structure. Here B^+_{dR} is the de Rham period ring of Fontaine, and this moduli is the B_{dR}-Grassmanian of the Berkeley notes [53].

b.- A perfect scheme $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}$ proper and perfectly finitely presented over $\mathrm{Spec}(k_{F_1})$ which parametrizes μ-bounded Witt-vector lattices with $\mathscr{G}$-structure. This is Zhu's Witt-vector Grassmanian [59], [5].

c.- A locally spatial diamond $\mathrm{Sht}_{(\mathscr{G},b,\mu),F_2^\diamond}$ partially proper over $F_2^\diamond$, parametrizing mixed-characteristic shtukas with G-structure that have relative position bounded by μ, and with level structure $\mathscr{G}(\mathbb{Z}_p)$. This is the moduli space of mixed-characteristic shtukas at hyperspecial level that appears in the Berkeley notes [53]. It comes endowed with a continuous $J_b(\mathbb{Q}_p)$-action.

d.- A perfect scheme $X^{\mathscr{G}}_{\leq\mu}(b)$ locally perfectly finitely presented over k_{F_2}, which on geometric points evaluates to affine Deligne-Lusztig sets of Rapoport [45]. This space also comes equipped with a continuous $J_b(\mathbb{Q}_p)$-action.

Fix an algebraically closed non-Archimedean field C over F_1 with ring of integers O_C and let k_C denote the residue field of O_C. In [1], Anschűtz constructs a map going from $\mathrm{Gr}^{G,\leq\mu}_{F_1^\diamond}(C,O_C)$ to $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}(k_C)$ which for now we denote sp_{Ans}. Before this work, the map was only known as a map of sets. Building on the work of Anschűtz we upgrade that specialization map to construct a specialization map of topological spaces $\mathrm{sp}_{\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O^\diamond_{F_1}}} : |\mathrm{Gr}^{G,\leq\mu}_{F_1^\diamond}| \to |\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}|$ making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Gr}^{G,\leq\mu}_{F_1^\diamond}(C,O_C) \arrow[r, "\iota"] \arrow[d, "\mathrm{sp}_{Ans}"] & \mid \mathrm{Gr}^{G,\leq\mu}_{F_1^\diamond} \mid \arrow[d, "\mathrm{sp}_{\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O^\diamond_{F_1}}}"] \\
\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}(k_C) \arrow[r, "\iota"] & \mid \mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}} \mid
\end{tikzcd}$$

Here ι associates to a $\mathrm{Spa}(C,O_C)$-valued point (k_C-valued point respectively) its underlying topological point. We prove the following properties about our specialization map.

Theorem 1. *a) The specialization map*

$$\mathrm{sp}_{\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O^\diamond_{F_1}}} : |\mathrm{Gr}^{G,\leq\mu}_{F_1^\diamond}| \to |\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}|$$

is a closed and spectral map of spectral topological spaces.

b) *Given a closed point* $x \in |\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}|$ *let* $T_x := \mathrm{sp}_{\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_{F_1}^{\diamondsuit}}}{}^{-1}(x)$, *then the interior* T_x° *of* T_x *in* $|\mathrm{Gr}^{G,\leq\mu}_{F_1{}^\diamondsuit}|$ *is a dense subset of* T_x.

c) T_x *and* T_x° *are non-empty and connected.*

Using a technique that we learned from reading [53] together with the work of Anschűtz, we construct a second specialization map but now its source is a moduli space of mixed-characteristic shtukas at hyperspecial level and the target is the affine Deligne-Lusztig variety associated to $(\mathscr{G}, b, \mu)$.

Theorem 2. a) *There is a continuous specialization map*

$$\mathrm{sp}_{\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_{F_2}}} : |\mathrm{Sht}_{(\mathscr{G},b,\mu),F_2^\diamondsuit}| \to |X^{\mathscr{G}}_{\leq\mu}(b)|,$$

this map is a specializing and spectral map of locally spectral topological spaces. It is a quotient map and it is $J_b(\mathbb{Q}_p)$*-equivariant.*

b) *Given a closed point* $x \in |X^{\mathscr{G}}_{\leq\mu}(b)|$ *let* $S_x = \mathrm{sp}_{\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_{F_2}}}{}^{-1}(x)$, *then the interior* S_x° *of* S_x *as a subspace of* $|\mathrm{Sht}_{(\mathscr{G},b,\mu),F_2^\diamondsuit}|$ *is dense in* S_x.

c) S_x *and* S_x° *are non-empty and connected.*

d) *The specialization map induces a* $J_b(\mathbb{Q}_p)$*-equivariant bijection of connected components*

$$\mathrm{sp}_{\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_{F_2}}} : \pi_0(\mathrm{Sht}_{(\mathscr{G},b,\mu),F_2^\diamondsuit}) \to \pi_0(X^{\mathscr{G}}_{\leq\mu}(b))$$

The work of Scholze and Weinstein identifies the diamond associated to moduli spaces of p-divisible groups as special instances of moduli spaces of mixed-characteristic shtukas (See [53] 24.3.5). Under this light, the last part of theorem 2 is a generalization of Theorem 5.1.5.(i) of [9] that describes the connected components of unramified Rapoport-Zink spaces at hyperspecial level. The study of the set of connected components of affine Deligne-Lusztig varieties had a lot of progress in the past 10 years. In the case of unramified groups at hyperspecial level the problem is very well understood, and the refer the reader to §3.4, to [41] theorem 1.1, to [9] theorem 1.1, or to [21] theorem 0.1, 0.2 for a concrete descriptions of these sets.

Another of our main results compares the preimages of the specialization map of Grassmanians to those of moduli spaces of shtukas. Before stating our result we mention a conjecture. The conjectural statement is philosophically aligned with Grothendieck-Messing's deformation theory of p-divisible groups and a weaker form of it is one of the key inputs in the proof of theorem 2. The statement is:

Conjecture 1. *If we let* $F_1 = F_2$, *then for a closed point* $x \in |X^{\mathscr{G}}_{\leq\mu}(b)|$ *there is a closed point* $y \in |\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_2}}|$ *such that* S_x° *considered as an open subdiamond of* $\mathrm{Sht}_{(\mathscr{G},b,\mu),F_2^\diamondsuit}$ *is isomorphic* *to T_y° when considered as an open subdiamond of $\mathrm{Gr}^{G,\leq\mu}_{F_2^\diamond}$. Here S_x and T_y are as in theorems 2 and 1 respectively.*

The weaker version that we are able to prove is as follows.

Theorem 3. *With the notation as in conjecture 1 there is a local model diagram*

$$\begin{tikzcd} & \mathcal{M} \arrow[dl, "g"'] \arrow[dr, "f"] & \\ S_x^\circ & & T_y^\circ \end{tikzcd}$$

and a v-sheaf in groups $\widehat{LG}$ such that the maps f and g are $\widehat{LG}$-bundles.

Before we describe our results on connected components at infinite level, we give a short summary of the theory of specialization of the first chapter and provide sketches of the proves of the main results of the second chapter.

Given a Tate Huber pair (A, A^+) with pseudo-uniformizer $\varpi \in A^+$ the specialization map $\mathrm{sp}_A : \mathrm{Spa}(A, A^+) \to \mathrm{Spec}(A^+/\varpi)$ assigns to $x \in \mathrm{Spa}(A, A^+)$ the prime ideal $\mathfrak{p}_x$ of those elements $a \in A^+$ for which $|a|_x < 1$. This is a continuous and closed map of spectral topological spaces and the construction is functorial in the category of Tate Huber pairs. The central idea of our theory is that, regardless of the definition, the specialization map for more general objects should also be functorial and should agree with the case of Tate Huber pairs. This desideratum naturally leads to defining the specialization map as the only map (if such a thing exists) that could be functorial. One is then forced to change perspective and to look for hypotheses that would prove that a functorial map exists and for conditions that would make this map unique.

The first question one needs to answer is what should the target and source of the specialization map be? The case of Tate Huber pairs may be a little bit misleading in that Tate Huber pairs come, by design, with a canonical "integral model". Namely, the integral model for $\mathrm{Spa}(A, A^+)$ is simply given by $\mathrm{Spa}(A^+, A^+)$. For more general spaces there is not a canonical "integral model" and one is forced to attach specialization maps to models rather than to the objects that one makes models of. In the case of Tate Huber pairs the specialization map can be extended to a map $\mathrm{sp}_{A^+} : \mathrm{Spa}(A^+, A^+) \to \mathrm{Spec}(A^+/\varpi)$ with the same formula. In the general case, one has to find an integral model for the diamond that one wishes to study. The integral models we will consider will be a subcategory of v-sheaves that satisfy some axioms.

An important result of the Berkeley notes proves that the $\diamond$-functor is a fully-faithful embedding of the category of characteristic p perfect schemes to Scholze's category of v-sheaves. Our observation is that, with the correct setup, this functor admits a right adjoint which we call suggestively the reduction functor. Moreover, we compute directly that if B is a topological ring over $\mathbb{Z}_p$ endowed with the I-adic topology for some finitely generated ideal I, then $(\mathrm{Spd}(B,B))^{\mathrm{red}}$ is given by the perfection of $\mathrm{Spec}(B/I)$. In particular, if (A,A^+) is a uniform Tate Huber pair, then $(\mathrm{Spd}(A^+,A^+))^{\mathrm{red}}$ is given by the perfection of $\mathrm{Spec}(A^+/\varpi)$. Recall that the underlying topological space of a scheme remains the same after taking its perfection. This suggests that one can define the target of our specialization to be the result of applying the reduction functor to the integral models that we want to associate a specialization map to. In general, the objects obtained in this way will not be perfect schemes but they will be what we call below scheme theoretic v-sheaves. These scheme theoretic v-sheaves come equipped with an underlying topological space that agrees with the Zariski topology whenever the sheaf is represented by a perfect scheme.

Once we have established the source and target, the next step is to construct the map. The key aspect that makes the specialization map for Tate Huber pairs functorial is that every map of Tate Huber pairs $\mathrm{Spa}(A,A^+) \to \mathrm{Spa}(B,B^+)$ automatically upgrades to a map of "integral models" $\mathrm{Spa}(A^+,A^+) \to \mathrm{Spa}(B^+,B^+)$. This motivates the following definition: given a v-sheaf $\mathcal{F}$, an affinoid perfectoid space $\mathrm{Spa}(A,A^+)$ and a map $\iota : \mathrm{Spa}(A,A^+) \to \mathcal{F}$, we say that $\mathcal{F}$ *formalizes* ι whenever it factors through a map $f : \mathrm{Spd}(A^+,A^+) \to \mathcal{F}$. We say that $\mathcal{F}$ is v-formalizing if for every ι as above there is a v-cover $g : \mathrm{Spa}(B,B^+) \to \mathrm{Spa}(A,A^+)$ such that $\mathcal{F}$ formalizes $\iota \circ g$. Given a v-formalizing sheaf $\mathcal{F}$ one can try to define the specialization map $\mathrm{sp}_{\mathcal{F}} : |\mathcal{F}| \to |\mathcal{F}^{\mathrm{red}}|$ so that for any "formalized" map $f : \mathrm{Spd}(A^+,A^+) \to \mathcal{F}$ the following diagram is commutative:

$$\begin{tikzcd}
\mid \mathrm{Spa}(A,A^+) \mid \arrow[r] \arrow[d, "\mathrm{sp}_A"] & \mid \mathrm{Spd}(A^+,A^+) \mid \arrow[r, "f"] & \mid \mathcal{F} \mid \arrow[d, "\mathrm{sp}_{\mathcal{F}}"] \\
\mid \mathrm{Spec}(A^+/\varpi) \mid \arrow[rr, "f^{\mathrm{red}}"] & & \mid \mathcal{F}^{\mathrm{red}} \mid
\end{tikzcd}$$

The recipe to compute the specialization map would then be the following: given $x \in |\mathcal{F}|$ for $\mathcal{F}$ v-formalizing we find an algebraically closed perfectoid field C and an open and bounded valuation ring C^+ together with a map $\iota_x : \mathrm{Spa}(C,C^+) \to \mathcal{F}$ such that the closed point of $\mathrm{Spa}(C,C^+)$ maps to x under ι_x. After replacing $\mathrm{Spa}(C,C^+)$ by a v-cover, we find a formalization $f_x : \mathrm{Spd}(C^+,C^+) \to \mathcal{F}$ of ι_x. We apply the reduction functor to f_x and obtain a map $f_x^{\mathrm{red}} : \mathrm{Spec}(C^+/\varpi) \to \mathcal{F}^{\mathrm{red}}$. Finally, we look at the topological image of the unique closed point of $\mathrm{Spec}(C^+/\varpi)$ under f_x^{red}. We define $\mathrm{sp}_{\mathcal{F}}(x)$ to be this image.

The natural question becomes whether or not this construction is well defined. The problem being that the map $\iota_x : \mathrm{Spa}(C,C^+) \to \mathcal{F}$ might have more than one formalization. The naive guess would be that this doesn't happen when $\mathcal{F}$ is separated as a v-sheaf. Unfortunately, this is false. At the heart of the problem is the following pathology: although $|\mathrm{Spa}(C,C^+)|$ is dense within $|\mathrm{Spa}(C^+,C^+)|$ it is not true that $|\mathrm{Spd}(C,C^+)|$ is dense within $|\mathrm{Spd}(C^+,C^+)|$ whenever the valuation ring C^+ has rank larger than 1.

Resolving this problem forces us to understand better the topological spaces of the form

$|\mathrm{Spd}(A, A^+)|$. To do so, we introduce what we call below the *olivine spectrum* of a Huber pair. The careful reader will notice that our work on the olivine spectrum of Huber pairs is a technification of Scholze and Weinstein's original approach to study perfect schemes as a full subcategory of the category of v-sheaves. Our work allow us to improve Scholze and Weinstein's full faithfulness result to the following statement:

Theorem 4. *Let Y be a perfect non-analytic adic space over $\mathbb{F}_p$ and let X be a pre-adic space over $\mathbb{Z}_p$. The natural map*

$$Hom_{_{PreAd}}(Y, X) \to Hom(Y^\diamondsuit, X^\diamondsuit)$$

is bijective. In particular, $\diamondsuit$ is fully faithful when restricted to the category of perfect non-analytic adic spaces over $\mathbb{F}_p$.

After this rather subtle and long topological detour, we manage to identify a stronger notion of separatedness that we call *formal separatedness*. The main feature of a formally separated v-sheaf $\mathcal{F}$ is that a map $\iota : \mathrm{Spa}(A, A^+) \to \mathcal{F}$ has at most one formalization (if any).

Combining the two inputs we say that a v-sheaf is specializing if it is v-formalizing and formally separated. We prove that specializing v-sheaves have a unique map that satisfies the commutative diagrams as above for any formalizable map. We prove that this specialization map is functorial in the full subcategory of specializing v-sheaves and that these specialization maps are continuous.

Although specializing v-sheaves produce the specialization maps that we are interested in, they are too general for practical purposes. For this reason, we focus our attention on a more restrictive class of v-sheaves that will have better behaved specialization maps. The central objects of our study is what we call below *kimberlites* (and *smelted kimberlites*). These will be specific kinds of specializing v-sheaves that satisfy other pleasant properties. For example, kimberlites come equipped with a good notion of "analytic locus" that is, by definition, an open subsheaf and a locally spatial diamond. The main advantage of kimberlites over more general specializing v-sheaves is that the specialization map of a kimberlite (when we restrict to the analytic locus) is a spectral maps of locally spectral spaces (i.e. continuous for the constructible topology). The author thinks of kimberlites as a first step towards the goal of formulating the notion of an "integral model" for diamonds. This is in the sense that if we wish to regard a v-sheaf as a "good" integral model for some (locally spatial) diamond, then it should at least satisfy the axioms to be a kimberlite.

Let us move on and discuss the content of the second chapter.

The construction of the specialization maps for the moduli spaces that we study follows from the general formalism that we discuss in the first chapter. To apply the theory one has to find a specializing v-sheaf "interpolating" the source and target of the desired specialization map. The candidates are already provided in the Berkeley notes [53]. More precisely, with the setup as in the beginning, Scholze and Weinstein describe what we call here the p-adic Beilinson-Drinfeld Grassmanians $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_{F_1}}$ as a v-sheaf over $O_{F_1}^{\diamondsuit}$ whose generic fiber is $\mathrm{Gr}^{G,\leq\mu}_{F_1^{\diamondsuit}}$. Also, they describe a v-sheaf $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_{F_2}}$ over $O_{F_2}^{\diamondsuit}$ whose generic fiber is $\mathrm{Sht}_{(\mathscr{G},b,\mu),F_2^{\diamondsuit}}$. We still call this v-sheaf the moduli space of mixed characteristic shtukas at hyperspecial level.

The proof that these v-sheaves are specializing uses all the machinery of modern p-adic Hodge theory as it is discussed in the Berkeley notes. Some key technical inputs are Kedlaya's work [29] and Anschűtz' work (theorem 1.2 of [1]) on extending vector bundles and torsors over the punctured spectrum of A_{inf}. Once we know these v-sheaves are specializing, our theory produces the specialization maps. With more work, we prove that these specializing v-sheaves are even nicer. Namely, we prove that p-adic Beilinson-Drinfeld Grassmanians are kimberlites, and that moduli spaces of shtukas at hyperspecial level are smelted kimberlites. We also prove the identities: $(\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_{F_1}})^{\mathrm{red}} = \mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}$ and $(\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_{F_2}})^{\mathrm{red}} = X^{\mathscr{G}}_{\leq\mu}(b)$ which tell us that the targets of the specialization maps that our formalism constructs are the desired ones.

After that work, the difficulty becomes to understand the preimages of the specialization map. To tackle this difficulty we introduce some theoretical tools. In the first chapter, to a kimberlite (or a smelted kimberlite) $\mathcal{F}$ and a chosen closed point $x \in |\mathcal{F}^{\mathrm{red}}|$ we attach *the tubular neighborhood* of $\mathcal{F}$ at x which we denote by $\widehat{\mathcal{F}}_{/x}$. Intuitively speaking, these tubular neighborhoods are the subsheaves of points that specialize to x. In general it is true that $\widehat{\mathcal{F}}_{/x}$ is a subsheaf of $\mathcal{F}$ and that $|\widehat{\mathcal{F}}_{/x}| \subseteq {\mathrm{sp}_{\mathcal{F}}}^{-1}(x)$, but the equality usually doesn't hold. One has to explore carefully the relation between these two sets. We identify a class of kimberlites (respectively smelted kimberlites), which we call *rich* kimberlites (respectively rich smelted kimberlites), for which tubular neighborhoods behave as nicely as possible. We prove that p-adic Beilinson-Drinfeld Grassmanians and moduli spaces of shtukas at hyperspecial level are rich kimberlites and rich smelted kimberlites, respectively. Being rich implies that $|\widehat{\mathcal{F}}_{/x}|$ is dense within ${\mathrm{sp}_{\mathcal{F}}}^{-1}(x)$, which proves the density part of theorems 1 and 2.

Once we know that these v-sheaves are rich kimberlites most of the work required to prove theorem 1 and theorem 2 is subsumed by our formalism. The last thing that remains to be proved is that the preimages of the specialization map are non-empty and connected. As we have briefly mentioned, one can apply theorem 3 to reduce the non-empty and connected part of theorem 2 to the similar claim of theorem 1.

To better understand the preimages of the specialization map in the case of theorem 1 we construct a "Demazure resolution" in the spirit of [53] §19.3. Contrary to the case of GL_n, for other reductive groups, the subset of dominant minuscule cocharacters doesn't generate the monoid of dominant cocharacters. This failure turns out to be a rather subtle matter and forces us to discuss what we call below "parahoric loop groups" for Chevalley groups. These groups are attached to points in the (Bruhat-Tits) apartment of G associated to T. They are subsheaves of the usual loop group given by the condition that their value on geometric points $\mathrm{Spa}(C^\sharp, C^{\sharp+})$ is precisely the corresponding parahoric subgroup of $G(B_{\mathrm{dR}})$ that the Bruhat-Tits theory attaches to the same point in the apartment and the (discrete valuation) period ring B^+_{dR}.

The construction that we discuss relies on a group-theoretic construction of Pappas and Zhu and on theorem 3.1 of [42]. The tradeoff to using parahoric loop groups is that now the "Schubert varieties" are indexed by elements of the Iwahori-Weyl group, and they can all be resolved using simple reflections in the affine Weyl group. Functoriality of the specialization map will allow us to reduce questions on the target of the resolution to questions on the source of the resolution. The v-sheaves that serve as source of this resolution, which we call Demazure kimberlites, are easier to understand.

Let us describe the content of the third chapter. In [46] Rapoport and Viehmann propose that there should be a theory of p-adic local Shimura varieties. They conjectured that there should exist towers of rigid-analytic spaces whose cohomology "understands" the local Langlands correspondence for general p-adic reductive groups. In this way, these towers of rigid-analytic varieties would "interact" with the local Langlands correspondence in a similar fashion to how Shimura varieties "interact" with the global Langlands correspondence. Moreover, they conjectured many properties and compatibilities that these towers should satisfy.

In the last decade, the theory of local Shimura varieties went through a drastic transformation with Scholze's introduction of perfectoid spaces and the theory of diamonds. In [53] Scholze and Weinstein construct the sought for towers of rigid analytic spaces and generalized them to what are now known as moduli spaces of p-adic shtukas. Moreover, since then, many of the expected properties and compatibilities for local Shimura varieties have been verified and generalized to moduli spaces of p-adic shtukas. The study of the geometry and cohomology of local Shimura varieties and moduli spaces of p-adic shtukas is still a very active area of research due to their connection to the local Langlands correspondence. The main aim of this chapter is to study the locally profinite set of connected components, and prove new cases of conjecture 4.26 in [46].

Let us recall the formalism of local Shimura varieties and moduli of p-adic shtukas. Local p-adic shtuka datum over $\mathbb{Q}_p$ is a triple $(G, [b], [\mu])$ where G is a reductive group over $\mathbb{Q}_p$, $[\mu]$ is a conjugacy class of geometric cocharacters $\mu : \mathbb{G}_m \to G$ and $[b]$ is an element of Kottwitz set $B(G, [\mu])$. Whenever $[\mu]$ is minuscule we say that $(G, [b], [\mu])$ is local Shimura datum. We let $E/\mathbb{Q}_p$ denote the reflex field of $[\mu]$. Associated to $(G, [b], [\mu])$ there is a tower of diamonds over $\mathrm{Spd}(\breve{E}, O_{\breve{E}})$, denoted $(\mathrm{Sht}_{G,[b],[\mu],\mathcal{K}})_{\mathcal{K}}$, where $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ ranges over compact subgroup of $G(\mathbb{Q}_p)$. Moreover, whenever $[\mu]$ is minuscule and $\mathcal{K}$ is a compact open subgroup, then $(\mathrm{Sht}_{G,[b],[\mu],\mathcal{K}})_{\mathcal{K}}$ is represented by the diamond associated to a unique smooth rigid-analytic space $\mathbb{M}_{\mathcal{K}}$ over $\breve{E}$. The tower $(\mathbb{M}_{\mathcal{K}})_{\mathcal{K}}$ is the local Shimura variety. Moreover if $\mathcal{K} = \mathscr{G}(\mathbb{Z}_p)$ for a reducitve group $\mathscr{G}$ over $\mathbb{Z}_p$ then $\mathrm{Sht}_{G,[b],[\mu],\mathcal{K}} = \mathrm{Sht}_{(\mathscr{G},b,\mu)}$, in the notation of theorem 2.

After basechange to a completed algebraic closure, each individual space $(\mathrm{Sht}_{G,[b],[\mu],\mathcal{K}} \times$

$\mathbb{C}_p)_{\mathcal{K}}$ comes equipped with continuous and commuting right actions by $J_b(\mathbb{Q}_p)$ and the Weil group W_E. Moreover, the tower receives a right action by the group $G(\mathbb{Q}_p)$ by using correspondances. When we let $\mathcal{K} = \{e\}$ we obtain the space at infinite level, denoted $\mathrm{Sht}_{G,[b],[\mu],\infty} \times \mathbb{C}_p$, which overall comes equipped with a continuous right action by $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E$.

This formalism is functorial on the group G in the following way. Whenever we are given a morphism of algebraic groups $f : G \to H$ over $\mathbb{Q}_p$ we obtain a morphism of towers

$$(\mathrm{Sht}_{G,[b],[\mu],\mathcal{K}} \times \mathbb{C}_p)_{\mathcal{K}} \to (\mathrm{Sht}_{H,[f(b)],[f\circ\mu],f(\mathcal{K})} \times \mathbb{C}_p)_{f(\mathcal{K})}$$

and these maps are equivariant with respect to the action induced by the map

$$G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E \to H(\mathbb{Q}_p) \times J_{f(b)}(\mathbb{Q}_p) \times W_E.$$

Since the actions are continuous the groups $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E$ act continuously on $\pi_0(\mathrm{Sht}_{G,[b],[\mu],\infty} \times \mathbb{C}_p)$ and our main theorem of chapter 3 describes explicitly this action whenever G is an unramified reductive group over $\mathbb{Q}_p$ and $([b],[\mu])$ is HN-irreducible. It is very likely that the methods of this thesis could be combined with those of [19] and [17] to remove the HN-irreducible condition. We do not pursue this generality.

Before stating this theorem we need to set more notation. Let $(G,[b],[\mu])$ be local p-adic shtuka datum with G an unramified reductive group over $\mathbb{Q}_p$. Let G^{der} denote the derived subgroup of G and G^{sc} denote the simply connected cover of G^{der}, let N denote the image of $G^{sc}(\mathbb{Q}_p)$ in $G(\mathbb{Q}_p)$ and let $G^\circ = G(\mathbb{Q}_p)/N$. This is a locally profinite topological group and it is the maximal abelian quotient of $G(\mathbb{Q}_p)$ when this later is considered as an abstract group.

Let $E \subseteq \mathbb{C}_p$ be the field of definition of $[\mu]$, let $\mathrm{Art}_E : W_E \to E^\times$ be Artin's reciprocity character from local class field theory. In §4 we associate to $[\mu]$ a continuous map of topological groups $Nm^\circ_{[\mu]} : E^\times \to G^\circ$ and we associate to $[b]$ a map $det^\circ : J_b(\mathbb{Q}_p) \to G^\circ$.

The general construction of $Nm^\circ_{[\mu]}$ and det° uses z-extensions and we do not review it in this introduction. Nevertheless, whenever $G^{sc} = G^{der}$ we can construct these maps as follows. In this case $G^\circ = G^{ab}(\mathbb{Q}_p)$ where G^{ab} is the co-center of G, which is an algebraic group of multiplicative type (or a torus). If we let $det : G \to G^{ab}$ be the quotient map we can consider the induced data $\mu^{ab} = det \circ [\mu]$ and $[b^{ab}] = [det(b)]$. Then $Nm^\circ_{[\mu]}$ can be defined as the following composition:

$$E^\times \xrightarrow{\mu^{ab}} G^{ab}(E) \xrightarrow{Nm^{G^{ab}}_{E/\mathbb{Q}_p}} G^{ab}(\mathbb{Q}_p) = G^\circ.$$

Here for a torus T over $\mathbb{Q}_p$, like G^{ab}, we are letting $Nm^T_{E/\mathbb{Q}_p} : T^{ab}(E) \to T^{ab}(\mathbb{Q}_p)$ denote the usual norm map

$$t \mapsto \prod_{\gamma \in Gal(E/\mathbb{Q}_p)} \gamma(t).$$

On the other hand, $det^\circ : J_b(\mathbb{Q}_p) \to G^{ab}(\mathbb{Q}_p)$ can be obtained as the composition $det =$

$j_{b^{ab}} \circ det_b$ where the maps $det_b : J_b(\mathbb{Q}_p) \to J_{b^{ab}}(\mathbb{Q}_p)$ and $j_{b^{ab}} : J_{b^{ab}}(\mathbb{Q}_p) \to G^\circ$ can be described as follows. The map det_b is obtained from functoriality of the formation of J_b, and $j_{b^{ab}}$ is the isomorphism $j_{b^{ab}} : J_{b^{ab}}(\mathbb{Q}_p) \cong G^{ab}(\mathbb{Q}_p)$ obtained from regarding $J_{b^{ab}}(\mathbb{Q}_p)$ and $G^{ab}(\mathbb{Q}_p)$ as subgroups of $G^{ab}(K_0)$ and exploiting that G^{ab} is commutative.

Theorem 5. *Let $(G, [b], [\mu])$ be local shtuka datum with G an unramified reductive group over $\mathbb{Q}_p$ and $([b], [\mu])$ HN-irreducible. The following hold:*

1. *The right $G(\mathbb{Q}_p)$ action on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathbb{C}_p)$ is trivial on $N = Im(G^{sc}(\mathbb{Q}_p))$ and the induced G°-action is simply-transitive.*

2. *If $s \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathbb{C}_p)$ and $j \in J_b(\mathbb{Q}_p)$ then*

$$s \cdot_{J_b(\mathbb{Q}_p)} j = s \cdot_{G^{ab}(\mathbb{Q}_p)} det^\circ(j^{-1}))$$

3. *If $s \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathbb{C}_p)$ and $\gamma \in W_E$ then*

$$s \cdot_{W_E} \gamma = s \cdot_{G^{ab}(\mathbb{Q}_p)} [Nm^\circ_{[\mu]} \circ Art_E(\gamma)].$$

Let us comment on previous results in the literature. Before a full theory of local Shimura varieties was available the main example of local Shimura varieties one could work with were the ones obtained as the generic fiber of a Rapoport-Zink space studied in [44]. The most celebrated examples of Rapoport-Zink spaces are of course the Lubin-Tate tower and the tower of covers of Drinfeld's upper half space. In [12] de Jong introduces his version of the fundamental group in rigid-analytic geometry to describe the Grothendieck-Messing period morphism. As an application of his theory of fundamental groups he computes the connected components of the Lubin-Tate tower for $\mathrm{GL}_n(\mathbb{Q}_p)$. In [58] Strauch computes by a very different method the connected components of the Lubin-Tate tower for $\mathrm{GL}_n(F)$ and an arbitrary finite extension F of $\mathbb{Q}_p$ (including ramification).

In [7] M. Chen constructs 0-dimensional local Shimura varieties and studies their geometry. These are the local Shimura varieties associated to tori. In a later paper [8] she constructs her "determinant" map and uses these 0-dimensional local Shimura varieties to describe connected components of Rapoport-Zink spaces of EL and PEL type associated to more general unramified reductive groups. We also use the determinant map, but in our case it is automatically constructed for us from the functoriality (with respect to group morphisms) of moduli spaces of p-adic shtukas. The central strategy of Chen's result builds on and improves the central strategy used by de Jong. Many steps in de Jong's original strategy fail or become technically more challenging when one passes from the Lubin-Tate tower to more general Rapoport-Zink spaces and M. Chen introduces many new ideas to tackle those cases. Two key inputs of Chen's work to the strategy is the use of her "generic" crystalline representations and her collaboration with Kisin and Viehmann on computing the connected components of affine Deligne-Lusztig varieties [9].

Our central strategy builds on the central strategy of de Jong and Chen, but the versatility of Scholze's theory of diamonds and the fully functorial construction of local Shimura varieties allow us to make many simplifications and streamline the proof. This is of course up to the fact that our arguments use Scholze's theory of diamonds rather than rigid analytic spaces. Our new main input to the central strategy is the use of specialization maps. To be able to use specialization maps in a rigorous way we had to develop a formalism that would allow us to use them. The details of this formalism are worked out in detail in the first two chapters. Originally, our formalism of specialization maps was developed to address a missing step in our efforts to adapt de Jong and Chen's strategy to the context of diamonds.

Let us sketch the central strategy to prove theorem 5. Once one knows that $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty}\times \mathbb{C}_p)$ is a right G°-torsor, computing the actions by W_E and $J_b(\mathbb{Q}_p)$ in terms of the G° action can be reduced to the tori case using functoriality, z-extensions and the determinant map. These uses mainly group theoretic methods and down to earth diagram chases. In the tori case the $J_b(\mathbb{Q}_p)$ action is easy to compute and the W_E action can be bootstrapped to an easier case as follows. For tori T, by the work of Kottwitz, we know that the set $B(T,\mu)$ has a unique element so that the data of b is redundant. We can consider the category of pairs (T,μ) where T is a torus over $\mathbb{Q}_p$ and μ is a geometric cocharacter whose field of definition is E. The construction of moduli spaces of shtukas is functorial with respect to this category. Moreover, this category has an initial object given by $(\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m),\mu_u)$ where

$$\mu_u : \mathbb{G}_m \to \mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m)_E$$

is the unique map of tori that on E-points is given by the formula

$$f \mapsto f \otimes_{\mathbb{Q}_p} f.$$

After more diagram chasing one can again reduce the tori case to the "universal" case. Finally, this case can be done explicitly using the theory of Lubin-Tate groups and their relation to class field theory. As we have mentioned, the tori case was already handled by M. Chen in [7], but for the convenience of the readers we recall the story in a different language.

Let us sketch how to prove that $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty}\times\mathbb{C}_p)$ is a G° torsor in the simplest case. For this let G be semisimple and simply connected. Our theorem then says that $\mathrm{Sht}_{G,b,[\mu],\infty}\times\mathbb{C}_p$ is connected.

The first step is to prove that $G(\mathbb{Q}_p)$ acts transitively on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty}\times\mathbb{C}_p)$. Using the Grothendieck-Messing period map one realizes that these is equivalent to proving that the b-admissible locus of Scholze's B_{dR}-Grassmanian is connected. This fact is a result of Hansen and Weinstein to which we give an alternative proof.

For the next step, let $x\in\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty}\times\mathbb{C}_p)$ and let $G_x\subseteq G(\mathbb{Q}_p)$ denote the stabilizer of x. Let $\mathcal{K}\subseteq G(\mathbb{Q}_p)$ be a hyperspecial subgroup of G. We claim that it is enough to prove that G_x is open and that $G(\mathbb{Q}_p)=\mathcal{K}\cdot G_x$. Indeed, $\mathcal{K}$ surjects onto $G(\mathbb{Q}_p)/G_x$ so that this space is discrete and compact therefore finite. By a theorem of Margulis [39], since we assumed G to be simply connected, the only open subgroup of finite index is the whole group so that $G_x=G(\mathbb{Q}_p)$.

The proof that G_x is open relies heavily on M. Chen's main technical result on her "generic" crystalline representations. To be able to apply her result in our context one uses that for suitable p-adic fields K, every crystalline representation is realized as a $\mathrm{Spd}(K, O_K)$-valued point in Scholze's B_{dR}-Grassmanian. For the convenience of the reader we include a discussion on how to think of crystalline representations as $\mathrm{Spd}(K, O_K)$-valued points. Finally, proving that $G(\mathbb{Q}_p) = \mathcal{K} \cdot G_x$ is equivalent to proving that $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathbb{C}_p$, the $\mathcal{K}$-level moduli space of shtukas, is connected. This is where our theory of specialization maps gets used. Indeed, theorem 2 proves that the specialization map identifies the connected components of moduli spaces of shtukas with the connected components of affine Deligne-Lusztig varieties. To conclude we only need to know that these varieties are connected.

Fortunately for us, the connected components of affine Deligne-Lusztig varieties are now very well understood by the work of many authors [9], [41] [22]. In the HN-irreducible case they can be identified with certain subset of $\pi_1(G)$. Since we assumed G to be simply connected $\pi_1(G) = \{e\}$ which finishes the sketch of the proof for the simply connected case. The central strategy used for general unramified groups G is not very different in spirit and only requires more patience.

Finally, let us comment on the organization of this thesis.

§1.1 We give a short review of the theory of diamonds, the v-topology and some facts about spectral topological spaces. We also review Scholze's $\lozenge$ functor that takes as input a pre-adic space over $\mathbb{Z}_p$ and returns as output a v-sheaf.

§1.2 We introduce and study what we call the olivine spectrum of a Huber pair (B, B^+) which we denote by $\mathrm{Spo}(B, B^+)$. As we have mentioned already, for a pre-adic space X over $\mathbb{Z}_p$ Scholze and Weinstein construct a surjective map of topological spaces $|X^\lozenge| \to |X|$. This map is a homeomorphism whenever X is analytic, but the map will not be injective whenever X is not analytic, and in pathological cases not even a quotient map. The olivine spectrum is a very concrete topological space that one can associate to any Huber pair without any mention to the theory of perfectoid spaces or diamonds. We can summarize the results as follows:

1. If (B, B^+) is any complete Huber pair over $\mathbb{Z}_p$ we construct a continuous and bijective map $|\mathrm{Spd}(B, B^+)| \to \mathrm{Spo}(B, B^+)$.
2. Whenever this map is a homeomorphism we say that (B, B^+) is *olivine*.
3. Being olivine can be verified locally in the usual topology of $\mathrm{Spa}(B, B^+)$ and it is compatible with rational localization.
4. Affinoid fields (i.e. (K, K^+) with K a field and K^+ an open bounded valuation subring) are olivine.
5. If (B, B^+) is uniform (i.e. B° is bounded) and B is a finite type B^+-algebra then it is olivine.

Using the olivine spectrum we prove theorem 4.

§1.3 We introduce and study a reduction functor that takes as input a small v-sheaf in the category of characteristic p perfectoid spaces, and returns a small v-sheaf in the category of perfect schemes in characteristic p. This functor generalizes the construction that assigns to a formal scheme topologically of finite type over $\mathbb{Z}_p$ the perfection of its reduced special fiber. To the author's knowledge, although this construction is simple, it had not been considered in the literature before. As we have mentioned, this reduction functor will construct the target of our specialization map.

§1.4 We develop the theoretical framework to study the specialization map. We introduce specializing v-sheaves, kimberlites, and smelted kimberlites. We introduce tubular neighborhoods and relate them with preimages of the specialization map. We define rich kimberlites which incorporate some "finiteness" conditions that are tailored to control the behavior of the preimages of the specialization map.

§2.1 We review the main geometric objects of modern p-adic Hodge theory. We review Kedlaya and Liu's theory of vector bundles on adic spaces, and the theorems of Kedlaya and Anschűtz' on extending vector bundles and G-torsors on the punctured spectrum of A_{inf}. One may think of Anschűtz' result as a statement over a point, and Scholze and Weinstein prove in [53] a small improvement to this theorem by considering what we call here a product of points. We review Scholze and Weinstein's proof with our application in mind.

§2.2 We study the specialization map for p-adic Beilinson-Drinfeld Grassmanians. We construct parahoric loop groups and construct Demazure kimberlites, which are the source of our "Demazure resolution". We prove that Demazure kimberlites are rich kimberlites which allows us to prove that p-adic Beilinson-Drinfeld Grassmanians are also rich kimberlites with non-empty connected tubular neighborhoods. We prove theorem 1.

§2.3 We study the specialization map for moduli spaces of mixed-characteristic shtukas. We prove that moduli spaces of mixed characteristic shtukas at hyperspecial level are rich smelted kimberlites. We prove theorems 2 and 3.

§3.1 Since the logic of chapter 3 is mostly independent of the previous chapters we reset the notation for that chapter. This is done in the first section.

§3.2 We recall the relation between crystalline representations, Scholze's theory of diamonds, and other geometric constructions that appear in modern p-adic Hodge theory. This part of the thesis is mainly expository, but we consider it important for the rest or the argument to have these relations in mind. We also include a discussion of Weil descent data and the action of $\underline{J_b(\mathbb{Q}_p)}$ since the author found some of the details in this part of the theory harder to grasp.

§3.3 We reprove M. Chen's results for tori. We do this for several reasons. On one hand, it was a very instructive exercise for the author to do this computation concretely, on the other hand the 0-dimensional local Shimura varieties that appear in Chen's work are constructed in a very different way. It is not clear to the author if proving that Chen's local Shimura varieties agree with Scholze and Weinstein's moduli spaces of shtukas is or not essentially equivalent to doing this computation.

§3.4 We provide the details of the proof of theorem 5.

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A note on the terminology

The author would like to use this paragraph to make a small comment on the terminology. Some of the terms introduced below come with a metaphor. The incorporation of these metaphors into the text is nothing but a playful manner in which the author decided to interact with the mineralogical history of the field. In particular, they shouldn't be taken seriously for any scientific or mathematical purposes.

The first term is the "olivine spectrum of a Huber pair". Olivine minerals are a series of mineralogical structures that can be found most commonly in mafic and ultramafic igneous rocks, they are characteristic by their green olive like color. During the formation of a diamond small minerals like olivine, garnet, and chromite among others get surrounded by a host diamond. When these minerals get included in diamonds their morphology changes to resemble the structure that is found in diamonds. Similarly, the olivine spectrum of a Huber pair is a very small variation of the usual adic spectrum that has a subtle diamond-like change.

The second term that deserves an explanation is "kimberlite". In mineralogy, kimberlites are hybrid rocks that are known to contain diamonds. The formation of diamonds happens in the depths of Earth and through geological processes, kimberlite magma pipes bring the diamonds to the surface. The interest in mining kimberlites comes from the hope of finding diamonds within. Similarly, the author thinks of kimberlites as a natural category for finding integral models of diamonds.

Chapter 1

Theoretical aspects of the specialization map

Throughout this thesis we assume that the reader is familiar with the basic theory of perfectoid spaces as discussed in ([53] §7) or ([51] §3). In most of our proofs we ignore the set-theoretic subtleties that arise from the theory, but we inherit the usage of the term "small" that is used to address such issues. We provide some indications on how to proceed when set-theoretic carefulness is absolutely necessary.

1.1 The v-topology

1.1.1 Diamonds and small v-sheaves

We let Perfd denote the category of perfectoid spaces and Perf the subcategory of perfectoid spaces in characteristic p. The following definition is taken from ([51] 7.8).

Definition 1.1.1. *Given a map of perfectoid spaces $f : Y \to X$ we say:*

1. *f is affinoid pro-étale if $Y = \mathrm{Spa}(S, S^+)$, $X = \mathrm{Spa}(R, R^+)$ and the map f is a small cofiltered limit of maps $f_i : \mathrm{Spa}(S_i, {S_i}^+) \to \mathrm{Spa}(R, R^+)$ where each f_i is étale.*

2. *f is pro-étale if for every $y \in Y$, there is an open neighborhood $V \subseteq Y$ containing y and an open $U \subseteq X$ satisfying $f(V) \subseteq U$ and $f|_V : V \to U$ is affinoid pro-étale.*

We can endow Perfd with two Grothendieck topologies, called the pro-étale topology and v-topology respectively, as follows:

Definition 1.1.2. *(See ([51] 8.1))*

1. *A family $\{f_i : Y_i \to X\}_{i\in I}$ of maps in* Perfd *is a cover for the pro-étale topology if each f_i is pro-étale and for every quasi-compact open $U \subseteq X$ there is a finite subset $I_U \subseteq I$ and quasi-compact open subsets $V_i \subseteq Y_i$ for all $i \in I_U$, such that $U = \bigcup f_i(V_i)$*

2. *A family* $\{f_i : Y_i \to X\}_{i\in I}$ *of maps in* Perfd *is a cover for the* v*-topology if for every quasi-compact open* $U \subseteq X$ *there is a finite subset* $I_U \subseteq I$ *and quasi-compact open subsets* $V_i \subseteq Y_i$ *for all* $i \in I_U$*, such that* $U = \bigcup f_i(V_i)$

Remark 1.1.3. *To make the pro-étale and* v *topologies useful, it is important to add the quasi-compactness hypothesis. Indeed, since open embeddings are étale the inclusion of points are pro-étale, but we do not want to consider the collection of inclusions of points as a cover.*

The following example of a cover for the v-topology will be used repeatedly.

Example 1.1.4. *Let* $\mathrm{Spa}(A, A^+)$ *be an affinoid perfectoid space* $\varpi \in A^+$ *and a choice of a pseudo-uniformizer, we consider the following construction. For every point* $x \in |\mathrm{Spa}(A, A^+)|$ *consider the inclusion of affinoid residue field*

$$\iota_x : \mathrm{Spa}(k(x), k(x)^+) \to \mathrm{Spa}(A, A^+).$$

Note that by ([49] 6.7) each $\mathrm{Spa}(k(x), k(x)^+)$ *is perfectoid. We now consider*

$$R^+ := \prod_{x\in|\mathrm{Spa}(A,A^+)|} k(x)^+$$

as a topological ring with the ϖ*-adic topology and let* $R = R^+[\frac{1}{\varpi}]$*. We have that* $\mathrm{Spa}(R, R^+)$ *is perfectoid and that the natural map* $\mathrm{Spa}(R, R^+) \to \mathrm{Spa}(A, A^+)$ *is a cover for the* v*-topology.*

Definition 1.1.5. *Given a set* I *and a collection of tuples* $\{(C_i, {C_i}^+), \varpi_i\}_{i\in I}$ *we construct an adic space* $\mathrm{Spa}(R, R^+)$*. Here each* C_i *is an algebraically closed non-Archimedean field, the* C_i^+ *are open and bounded valuation subrings of* C_i*, and* ϖ_i *is a choice of pseudo-uniformizer. We let* $R^+ := \prod_{i\in I} C_i^+$*, we let* $\varpi = (\varpi_i)_{i\in I}$*, we endow* R^+ *with the* ϖ*-adic topology and we let* $R := R^+[\frac{1}{\varpi}]$*. Any space constructed in this way will be called a product of points.*

Remark 1.1.6. *We point out that different choices of pseudo-uniformizers* $(\varpi_i)_{i\in I}$ *will give rise to different adic spaces. Indeed, in general* $R \subseteq \prod_{i\in I} C_i$ *but if* I *is infinite this is a proper inclusion and the image of this inclusion depends of the choice of pseudo-uniformizers.*

Example 1.1.4 proves that products of points form a basis for the v-topology in the category of perfectoid spaces. Recall the notion of totally disconnected spaces.

Definition 1.1.7. *(See [51] 7.1, 7.15, 7.5) An affinoid perfectoid space* $\mathrm{Spa}(R, R^+)$ *is totally disconnected if it splits every open cover. Moreover, it is strictly totally disconnected if it splits every étale cover.*

We have the following useful criterion:

Proposition 1.1.8. *(See [51] 7.3, 7.16, 11.27) Let* Y *be an affinoid perfectoid space.* Y *is represented by a strictly totally disconnected space if and only if every connected component of* Y *is represented by* $\mathrm{Spa}(C, C^+)$ *for* C *an algebraically closed field and* C^+ *an open and bounded valuation subring.*

Proposition 1.1.9. *Product of points as in definition 1.1.5 are strictly totally disconnected perfectoid space.*

Proof. Take $R^+ := \prod_{i\in I} C_i^+$ and pseudo-uniformizers $\varpi = (\varpi_i)_{i\in I}$ as in definition 1.1.5. The closed-opens subsets of $\mathrm{Spa}(R, R^+)$ are given by idempotents in R^+, which in turn are given by subsets of I. A connected component $\pi \in \pi_0(\mathrm{Spa}(R, R^+))$ is computed by intersecting $\bigcap_{U\in\mathcal{U}} U$ for some ultrafilter and it is a Zariski closed subsets cut out by the ideal, $I_\pi = \langle 1_V \rangle_{V\notin\mathcal{U}}$, where the idempotents are indexed by the sets that do not belong to the ultrafilter. To compute the structure sheaf of this connected component we have to consider the ϖ-completion of R^+/I_π. Let $V = R^+/I_\pi$ and V' be the completion of V with respect to ϖ.

To prove that $\mathrm{Spa}(R, R^+)$ is a strictly totally disconnected perfectoid space it is enough, by proposition 1.1.8, to prove that V' is a valuation ring with algebraically closed fraction field. In general, if W is a valuation ring with algebraically closed fraction field and if $a \in W$ is not a unit, then the (a)-adic completion of W is also a valuation ring with algebraically closed fraction field. Applying this reasoning to V and V', we see that it is enough to prove V is a valuation ring with algebraically closed fraction field.

To prove that V is a domain take two elements in $v_1, v_2 \in R^+$ with $v_1 \cdot v_2 = 0$. If we let $I_j \subseteq I$ with $j \in \{1, 2\}$ be the subsets of $i \in I$ such that $v_j = 0$ in C_i^+ then $I_1 \cup I_2 = I$ and one of I_1 or $I_2 \setminus I_1$ is in the ultrafilter, this implies that one of v_1 or v_2 equals 0 in V. Take an element of $v \in Frac(V)$, this element may be represented by an element of $\prod_{i\in I} C_i$. Since each entry of the product defining R^+ is a valuation ring one of the sets $\{i \in I \mid v_i \in C_i^+\}$ or $\{i \in I \mid v_i^{-1} \in C_i^+\}$ is in the ultrafilter, this implies $v \in V$ or $v^{-1} \in V$ and that V is a valuation ring. One can prove in a similar way that $Frac(V) = \prod_{i\in I} C_i/I_\pi$. In particular, it is an ultraproduct of algebraically closed fields, so $Frac(V)$ is algebraically closed. $\square$

Scholze proves that the v-topology (and consequently the pro-étale topology) on Perfd is subcanonical ([51] 8.6). To simplify notation, we denote a perfectoid space and the sheaf it represents with the same letter. In case we need to make a distinction, whenever X is a perfectoid space the sheaf it represents will be denoted by h_X. From now on we will focus most of our attention to the site Perf endowed with either the pro-étale or the v-topology. Let us recall Scholze's category of diamonds.

Definition 1.1.10. *(See [51] 11.1) A pro-étale sheaf Y on* Perf *is a diamond if it can be written as X/R where X and R are representable by perfectoid spaces and $R \subseteq X \times X$ is an equivalence relation for which the two projections to X are pro-étale maps of perfectoid spaces.*

Given a diamond Y we can associate to it a topological space, denoted $|Y|$, as follows:

Definition 1.1.11. *We say that a map $p : \mathrm{Spa}(K, K^+) \to Y$ is a point if K is a perfectoid field in characteristic p and K^+ is an open and bounded valuation subring of K. Two points $p_i : \mathrm{Spa}(K_i, {K_i}^+) \to Y$, $i \in \{1, 2\}$, are equivalent if there is a third point $p_3 : \mathrm{Spa}(K_3, {K_3}^+) \to$*

Y, and surjective maps $q_i : \mathrm{Spa}(K_3, {K_3}^+) \to \mathrm{Spa}(K_i, {K_i}^+)$ making the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Spa}(K_1, {K_1}^+) \arrow[dr, "p_1"] & \\
\mathrm{Spa}(K_3, {K_3}^+) \arrow[ur, "q_1"] \arrow[rr, "p_3"] \arrow[dr, "q_2"'] & & Y \\
 & \mathrm{Spa}(K_2, {K_2}^+) \arrow[ur, "p_2"'] &
\end{tikzcd}$$

We let $|Y|$ denote the set of equivalence classes of points of Y.

Scholze proves that if Y has a presentation X/R, then there is canonical bijection of sets between $|Y|$ and $|X|/|R|$ (where $|X|$ and $|R|$ are the topological space corresponding to the perfectoid spaces X and R). This gives a surjective map $|X| \to |Y|$ and we give $|Y|$ the quotient topology for such a map.

Proposition 1.1.12. *(See [51] 11.13) Let Y be a diamond. The topology on $|Y|$ is independent of the presentation of Y as a quotient $Y = X/R$, with X and R perfectoid spaces.*

We remark that if X is a perfectoid space, then the sheaf h_X represented by X is a diamond and that $|h_X|$ is canonically homeomorphic to $|X|$.

We refer to sheaves on Perf for the v-topology as v-sheaves and we say that a v-sheaf X is small if it admits a surjective map from a representable sheaf. This is a set theoretic condition.

Proposition 1.1.13. *(See [51] 11.9) Every diamond is a small v-sheaf.*

Recall that any Grothendieck site has an intrinsic notion of quasi-compactness. Quasi-compact v-sheaves are other important examples of small v-sheaves.

We denote by $\widetilde{\mathrm{Perf}}$ the category of small v-sheaves, it may be constructed as follows. Given a cut-off cardinal κ (see [51] §4 and §8 for details) we denote by Perf_κ the category of κ-small perfectoid spaces in characteristic p and by $\widetilde{\mathrm{Perf}}_\kappa$ the topos of sheaves for the v-topology on this category. Objects in this topos will be called κ-small v-sheaves. We have natural fully-faithful embeddings $\widetilde{\mathrm{Perf}}_\kappa \subseteq \widetilde{\mathrm{Perf}}_\lambda$ for $\kappa < \lambda$ and we define $\widetilde{\mathrm{Perf}} = \bigcup_\kappa \widetilde{\mathrm{Perf}}_\kappa$ as a big filtered colimit over all cut-off cardinals κ.

Scholze associates to any small v-sheaf a topological space. The definition is almost identical to 1.1.11, the key point being that if $X \to Y$ is a map of small v-sheaves with X a diamond then $R = X \times_Y X$ is also a diamond and $Y = X/R$ ([51] 12.3). Scholze then defines $|Y|$ as $|X|/|R|$ with the quotient topology.

Proposition 1.1.14. *(See [51] 12.7) Let Y be a small v-sheaf. The set of equivalence classes of points of Y is in canonical bijection with $|X|/|R|$ for any presentation $Y = X/R$ with X and R diamonds. Moreover, the topology induced this way is independent of the presentation.*

Given a topological space T we can consider a presheaf on Perf, denoted $\underline{T}$, defined as

$$\underline{T}(R, R^+) = \{f : |\mathrm{Spa}(R, R^+)| \to T \mid f \textit{ is continuous}\}$$

This forms a v-sheaf but we warn the reader that it might not be small. There is a natural transformation:

$$X \to \underline{|X|}$$

A morphism of small v-sheaves $j : U \to X$ is said to be open if it is relatively representable in perfectoid spaces and after basechange by a perfectoid space it becomes an open embedding of perfectoid spaces. The following proposition shows that this is a purely topological notion.

Proposition 1.1.15. *(See [51] 11.15 and 12.9) Let Y be a small v-sheaf and let $|V|' \subseteq |Y|$ be an open subset. Define V as the Cartesian product:*

$$\begin{tikzcd} V \arrow[r] \arrow[d] & \underline{|V|'} \arrow[d] \\ Y \arrow[r] & \underline{|Y|} \end{tikzcd}$$

The following assertions hold:

1. *The map $V \to Y$ is an open embedding of small v-sheaves.*

2. *The induced map $|V| \to |Y|$ is an open embedding of topological spaces and factors through a homeomorphism to $|V|'$.*

3. *Every open embedding of small v-sheaves is isomorphic to one constructed in this way.*

4. *If Y is a diamond then V is also a diamond.*

The concept of closed immersion is a little more subtle. It is not a purely topological condition in the sense that closed subsheaves of $\mathcal{F}$ are not in one to one bijection with closed subsets of $|\mathcal{F}|$.

Definition 1.1.16. *(See [51] 10.7, 10.11, 5.6) A map of sheaves $\mathcal{F} \to \mathcal{G}$ is a closed immersion if for every $X = \mathrm{Spa}(R, R^+)$ a strictly totally disconnected space and a map $X \to \mathcal{G}$ the pullback $X \times_{\mathcal{F}} \mathcal{G} \subseteq X$ is representable by a closed immersion.*

The following result of Johanes Anschűtz, João Lourenço and Timo Richarz, that will appear in [26], characterizes closed immersions.

Proposition 1.1.17. *(See [26]) For a v-sheaf $\mathcal{F}$ we say a subset $X \subseteq |\mathcal{F}|$ is weakly generalizing if for any geometric point $f : \mathrm{Spa}(C, C^+) \to \mathcal{F}$ we have that $f^{-1}(X) \subseteq |\mathrm{Spa}(C, C^+)|$ is stable under generization. For any v-sheaf $\mathcal{F}$ the rule*

$$X \mapsto \mathcal{F} \times_{\underline{|\mathcal{F}|}} \underline{X} \subseteq \mathcal{F}$$

gives a bijection between weakly generalizing closed subsets of $|\mathcal{F}|$ and closed subsheaves of $\mathcal{F}$.

1.1.2 Spectral spaces and locally spatial diamonds

The category of diamonds is too general for some purposes and one can construct many "pathological" examples of diamonds that do not arise from an algebro-geometric context. To control this flexibility Scholze considers some restrictions on the underlying topological space of a diamond.

We begin by recalling the basic theory of spectral topological spaces. This material is taken from section §2 of [51] where most of the proves can be found.

Definition 1.1.18. *For topological spaces S, T and a continuous map $f : S \to T$ we say that:*

1. *T is spectral if it is quasi-compact, quasi-separated, and it has a basis of open neighborhoods stable under intersection that consists of quasi-compact and quasi-separated subsets.*
2. *T is locally spectral if it admits an open cover by spectral spaces.*
3. *f is a spectral map of spectral spaces if S and T is are spectral and f is quasi-compact.*
4. *f is a spectral map of locally spectral spaces if S and T are locally spectral and for every quasi-compact open $U \subseteq S$ and quasi-compact open $V \subseteq T$ with $f(U) \subseteq V$ the function $f|_U : U \to V$ is spectral.*

Theorem 1.1.19. *(Hochsteter) For a topological space T the following conditions are equivalent:*

1. *T is spectral.*
2. *T is homeomorphic to the spectrum of a ring.*
3. *T is a projective limit of finite T_0 topological spaces.*

Moreover, the category of spectral topological spaces with spectral maps is equivalent to the pro-category of finite T_0 topological spaces.

Given a spectral space T, we say that a subset S is constructible if it lies in the Boolean algebra formed by quasi-compact open subsets of T. For a locally spectral space T, a subset S is constructible if for every quasi-compact open subset $U \subseteq T$ the subset $S \cap U$ is constructible in U. The patch (or constructible) topology on T is the one in which constructible subsets form a basis for the topology. A spectral space is Hausdorff and profinite for its patch topology and a locally spectral space is Hausdorff and locally profinite for the patch topology.

Proposition 1.1.20. *A continuous map of locally spectral spaces $f : S \to T$ is spectral if and only if it is continuous for the patch topology.*

Definition 1.1.21.

1. *A map of topological spaces $f : S \to T$ is generalizing if for elements $t_1, t_2 \in T$ and $s_1 \in S$ such that $f(s_1) = t_1$ and t_2 generalizes t_1, there exists an element s_2 generalizing s_1 with $f(s_2) = t_2$.*

2. *A map of topological spaces $f : S \to T$ is specializing if for elements $t_1, t_2 \in T$ and $s_1 \in S$ such that $f(s_1) = t_1$ and t_2 specializes from t_1, there exists an element s_2 specializing from s_1 with $f(s_2) = t_2$.*

For a locally spectral space T we say that a subset is pro-constructible if it is closed for the patch topology, or equivalently if it is an arbitrary intersection of constructible subsets. The following will be really useful for our purposes.

Proposition 1.1.22. *(See [51] 2.4) Let T be a spectral space and $S \subseteq T$ a pro-constructible subset. The closure $\overline{S}$ of S in T consists of the points that specialize from a point in S.*

Corollary 1.1.23. *Let $f : S \to T$ be a spectral map of spectral spaces. If f is specializing then it is also a closed map.*

We warn the reader that the analogue of 1.1.23 for locally spectral spaces does not hold.

Proposition 1.1.24. *(See [51] 2.5) Let $f : S \to T$ be a spectral map of spectral topological spaces. Assume f is surjective and generalizing, then it is a quotient map.*

One can think of spectral spaces as the topological spaces that arise from an algebro-geometric situation. For this reason we will restrict our attention to diamonds that have this behavior.

Definition 1.1.25. *(See [51] 11.17) Let X be a diamond. We say that X is a spatial diamond if it is quasi-compact, quasi-separated and $|X|$ has a basis of open neighborhoods of the form $|U|$ where $U \subseteq X$ is a quasi-compact open embedding. We say that X is locally spatial if it has an open cover by spatial diamonds.*

As promised, the topology of spatial diamonds is spectral. Nevertheless, we remark that a diamond that has a spectral underlying topological space might not necessarily be spatial since the quasi-compactness and quasi-separatedness conditions of definition 1.1.25 are imposed on the topos-theoretic sense.

Proposition 1.1.26. *(See [51] 11.18, 11.19) Let X and Y a be locally spatial diamonds and $f : X \to Y$ a morphism of v-sheaves. The following assertions hold:*

1. *$|X|$ is a locally spectral topological space.*
2. *Any open subfunctor $U \subseteq X$ is a locally spatial diamond.*
3. *$|X|$ is quasi-compact (respectively quasi-separated) as a topological space if and only if X is quasi-compact (respectively quasi-separated) as a v-sheaf.*
4. *The topological map $|f|$ is spectral and generalizing. In particular, if $|X|$ is quasi-compact and $|f|$ is surjective then by proposition 1.1.24 it is also a quotient map.*

1.1.3 Pre-adic spaces as v-sheaves

The theory of diamonds is mainly of "analytic" nature. On the other hand, we will need to consider some spaces that have a scheme-theoretic and formal-scheme-theoretic flavor instead. The category of v-sheaves allows us to consider these three types of spaces at the same time. In what follows, we show (following the Berkeley notes) how to consider any pre-adic space over $\mathbb{Z}_p$ as a v-sheaf. Interestingly, this functor is by construction far from being fully-faithful, but we will justify below why the new morphisms are mostly of analytic nature.

Let us give a quick recollection of the appendix to lecture 3 of [53]. Let Caff^{op} denote the opposite category to the category of complete Huber pairs. This category can be regarded as a site when we consider the topology generated by rational covers. Although the topology in this site is not subcanonical any Huber pair $(A, A^+) \in \mathrm{Caff}^{op}$ defines a sheaf $\mathrm{Spa}(A, A^+)^Y : \mathrm{Caff}^{op} \to \mathrm{Sets}$ by taking sheafification of the functor $(B, B^+) \mapsto Hom_{\mathrm{Caff}}((A, A^+), (B, B^+))$. Scholze and Weinstein define the category of Yoneda-adic spaces as sheaves $\mathcal{F} : \mathrm{Caff}^{op} \to \mathrm{Sets}$ that are locally isomorphic to $\mathrm{Spa}(A, A^+)^Y$ for a suitable notion of open immersion of sheaves. Recall the category $(V)^{ind}$ whose objects are triples $(X, \mathcal{O}_X^{ind}, (|\cdot(x)|)_{x\in X})$ where X is a topological space, $\mathcal{O}_X^{ind}$ is a sheaf of ind-topological rings and $|\cdot(x)|$ is an equivalence class of valuations on $\mathcal{O}_{X,x}$. For a Huber pair (A, A^+) Scholze and Weinstein define $\mathrm{Spa}^{ind}(A, A^+)$ to be the object in (V^{ind}) with underlying topological space $\mathrm{Spa}(A, A^+)$ and with $\mathcal{O}_X^{ind}$ defined as the sheafification of the structure presheaf of $\mathrm{Spa}(A, A^+)$ in the category of ind-topological rings. A pre-adic space is an object $X \in (V^{ind})$ that is locally of the form $\mathrm{Spa}^{ind}(A, A^+)$.

Proposition 1.1.27. *(See [53] 3.5.3) For a pre-adic space X, the functor*

$$h_X = ((A, A^+) \mapsto Hom_{(V^{ind})}(\mathrm{Spa}^{ind}(A, A^+), X))$$

is a Yoneda adic space. The functor $X \mapsto h_X$ is an equivalence of categories between pre-adic spaces and Yoneda-adic spaces. This equivalence preserves the notions of open immersions and the functor associated to $\mathrm{Spa}^{ind}(A, A^+)$ gets identified with $\mathrm{Spa}^Y(A, A^+)$.

Certain things are easier to think in one perspective than in the other. An important aspect is that every pre-adic space X has an underlying topological space, and we can define the open analytic locus $|X|^{an}$ and the non-analytic locus $|X|^{na}$ in the naive way. That is, a point $x \in |X|$ is analytic if for every open affinoid $x \in \mathrm{Spa}^{ind}(A, A^+) \subseteq |X|$ (equivalently one affinoid) x is analytic in $\mathrm{Spa}(A, A^+)$.

Proposition 1.1.28. *Given a pre-adic space X there is a reduced non-analytic adic space X^{na} and a map $X^{na} \to X$ which is final in the category of maps $Y \to X$ with Y a reduced non-analytic adic space. Moreover, the map $|X^{na}| \to |X|^{na}$ is a homeomorphism.*

Proof. In the affinoid case we have that $\mathrm{Spa}^{ind}(A, A^+)^{na}$ is given by

$$\mathrm{Spa}(A/A^{\circ\circ} \cdot A, A^+/A^{\circ\circ} \cdot A^+).$$

Observe that since $A/A^{\circ\circ} \cdot A$ is discrete it is sheafy. Moreover if (B, B^+) has the discrete topology then $Hom(\mathrm{Spa}^{ind}(B, B^+)), \mathrm{Spa}^{ind}(A, A^+))$ are given by maps of Huber pairs $(A, A^+) \to (B, B^+)$ and all topological nilpotents must map to 0 in B proving the universal property. The claim of topological spaces is clear.

For general pre-adic space X we define X^{na} to have underlying topological space $|X|^{na}$ and if $V \subseteq |X^{na}|$ is of the form $U \cap |X|^{na}$ for $U \subseteq |X|$ open and of the form $U = \mathrm{Spa}^{ind}(A, A^+)$ we let

$$\mathcal{O}^{ind}_{X^{na}}(V) := \mathcal{O}^{ind}_X(U)/A^{\circ\circ} \cdot \mathcal{O}^{ind}_X(U).$$

Since the construction $A \mapsto A/A^{\circ\circ}$ is compatible with rational localization $\mathcal{O}^{ind}_{X^{na}}(V)$ is well-defined and glues to a sheaf of ind-topological rings on X^{na}. Moreover, locally the ind-topological rings are constant because $A/A^{\circ\circ}$ is sheafy. This implies X^{na} is an adic space. □

Definition 1.1.29. *1. We define the presheaf $\mathbb{Z}_p^\Diamond$ on $Perf$ as the moduli of untilts, more precisely:*

$$\mathbb{Z}_p^\Diamond(Y) = \{(Y^\sharp, \iota)\}/\cong$$

Where $Y^\sharp$ is a perfectoid space in Perfd *and $\iota : (Y^\sharp)^\flat \to Y$ is an isomorphism of perfectoid spaces in characteristic p.*

2. Given a pre-adic space $X/\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$ we define the presheaf $X^\Diamond$ on $Perf$ as:

$$X^\Diamond(Y) = \{(Y^\sharp, \iota, f)\}/\cong$$

Where $Y^\sharp \in$ Perfd, $\iota : (Y^\sharp)^\flat \to Y$ is an isomorphism of perfectoid spaces in characteristic p, and $f : Y^\sharp \to X$ is a morphism of pre-adic spaces.

Notice that there is a canonical morphism $X^\Diamond \to \mathbb{Z}_p^\Diamond$ given by forgetting the last entry of data.

Proposition 1.1.30. *(See [53] 18.1.1) For any pre-adic space X (not necessarily analytic) over $\mathbb{Z}_p$, the presheaf $X^\Diamond$ is a small v-sheaf.*

From now on, given a Huber pair (A, A^+) we will denote the v-sheaf $(\mathrm{Spa}(A, A^+)^Y)^\diamond$ by $\mathrm{Spd}(A, A^+)$ and we will also drop the decoration $(\cdot)^Y$ when referring to affinoid pre-adic spaces. If R is a base ring whose underlying topology (and ring of integral elements) is understood from the context we will abbreviate $\mathrm{Spd}(R, R^+)$ by $R^\diamond$. For example, $\mathbb{F}_p^\diamond$, $\mathbb{Z}_p^\diamond$, $\mathbb{Q}_p^\diamond$, etc.

Proposition 1.1.31. *Let us collect some facts about $\diamond$, that are either in the literature or are easy to prove:*

1. *For any perfectoid space X we have that $X^\diamond \cong h_{X^\flat}$ as v-sheaves. (See [51] 15.2).*
2. *For any analytic pre-adic space X over $\mathbb{Z}_p$, the functor $X^\diamond$ is a locally spatial diamond and $|X^\diamond| \cong |X|$. (See [51] 15.6).*
3. *For any pre-adic space over $\mathbb{Z}_p$ there is a surjective map of topological spaces $|X^\diamond| \to |X|$ (See [53] 18.2.2).*
4. *If $\mathrm{PreAd}_{\mathbb{Z}_p}$ denotes the category of pre-adic spaces over $\mathbb{Z}_p$ then $\diamond : \mathrm{PreAd}_{\mathbb{Z}_p} \to \widetilde{\mathrm{Perf}}$ commutes with limits and colimits. More precisely, if X_i is a family of pre-adic spaces indexed by a small category I and the functor $\varinjlim_{i\in I} X_i$ (respectively $\varprojlim_{i\in I} X_i$) is represented by a pre-adic space X then $X^\diamond = \varinjlim_{i\in I} X_i^\diamond$ (respectively $X^\diamond = \varprojlim_{i\in I} X_i^\diamond$). Indeed, both computations are done in the category of sheaves of a Grothendieck site. The only difference in the computations is the topology that one has to use to sheafify. But if a colimit is represented by a pre-adic space by proposition 1.1.30 it is already a v-sheaf.*
5. *For any complete Huber pair (B, B^+) over $\mathbb{Z}_p$ the v-sheaf $\mathrm{Spd}(B, B^+)$ is separated over $\mathbb{Z}_p^\diamond$. Indeed, the basechange of the diagonal map $\mathrm{Spd}(B, B^+) \to \mathrm{Spd}(B, B^+) \times_{\mathbb{Z}_p^\diamond} \mathrm{Spd}(B, B^+)$ by any map $\mathrm{Spa}(R, R^+) \to \mathrm{Spd}(B, B^+) \times_{\mathbb{Z}_p^\diamond} \mathrm{Spd}(B, B^+)$ is given by the Zariski closed immersion defined by the ideal $R^\sharp \cdot I_\Delta$ with I_Δ the image of $\ker(B \otimes_{\mathbb{Z}_p} B \to B)$ in $R^\sharp$.*
6. *For any pre-adic space X the map of v-sheaves $(X^{na})^\diamond \to X^\diamond$ is a closed immersion of v-sheaves and $X^\diamond \setminus (X^{na})^\diamond = (X^{an})^\diamond$. Indeed, this can be verified locally so we may assume $X = \mathrm{Spa}(A, A^+)$. If $Y = \mathrm{Spa}(R, R^+)$ is a strictly totally disconnected space then $Y \times_{X^\diamond} (X^{na})^\diamond$ is a Zariski closed immersion defined by the image ideal of $A^{\circ\circ}$ in $R^\sharp$.*

1.2 The olivine spectrum

As we will see below, given a pre-adic space X over $\mathbb{Z}_p$ the map $|X^\diamond| \to |X|$ of remark 1.1.31 will usually not be injective when X has non-analytic points. Although the map is always surjective, it might not be a quotient map in pathological cases. To develop our theory of specialization map we need better understanding of the topological spaces of the form

$|\mathrm{Spd}(A, A)|$ for I-adic rings A over $\mathbb{Z}_p$. To tackle this difficulty we introduce below what we call the *olivine spectrum* of a Huber pair, which is a very small variation of Huber's adic spectrum. The interest in studying the olivine spectrum is that if (B, B^+) is a complete Huber pair over $\mathbb{Z}_p$ subject to some mild "finiteness" conditions then the topological space $|\mathrm{Spd}(B, B^+)|$ is homeomorphic to the olivine spectrum of (B, B^+).

For the following fix (B, B^+) a Huber pair (not necessarily over $\mathbb{Z}_p$ and not necessarily complete).

1.2.1 Review, terminology and conventions

We assume the reader to be familiar with the construction of Huber's adic spectrum, $\mathrm{Spa}(B, B^+)$, but we review some key aspects and definitions. We also fix some terminology.

1. Given $x \in \mathrm{Spa}(B, B^+)$ we define the support $supp(x) \subseteq B$ as the set of elements $b \in B$ for which $|b|_x = 0$. This is a prime ideal of B.

2. We say that a point $x \in \mathrm{Spa}(B, B^+)$ is non-analytic if $supp(x)$ is an open ideal of B, we say it is analytic otherwise.

3. Given an equivalence class of valuations on B, say represented by $|\cdot|_x : B \to \Gamma_x \cup \{0\}$, and a convex subgroup $H \subseteq \Gamma_x$, we define a second equivalence class of valuations represented by $|\cdot|_y : B \to (\Gamma_x/H) \cup \{0\}$ with $|b|_y = |b|_x + H \in \Gamma_x/H$ when $|b|_x \neq 0$ and $|b|_y = 0$ when $|b|_x = 0$. Any equivalence class of valuations constructed in this way is called a *vertical generization* of x.

4. Given a complete Huber pair (B, B^+) and a point $x \in \mathrm{Spa}(B, B^+)$ there is a residue field map of complete Huber pairs $\iota_x^* : (B, B^+) \to (K_x, {K_x}^+)$. In this case K_x is either a discrete field or a complete non-Archimedean field. In both cases, K_x^+ is an open and bounded valuation subring of K_x. The induced map $\iota_x : \mathrm{Spa}(K_x, {K_x}^+) \to \mathrm{Spa}(B, B^+)$ is a homeomorphism onto the subspace of $\mathrm{Spa}(B, B^+)$ consisting of continuous vertical generizations of x. The map satisfies the following universal property: For any map of complete Huber pairs $f^* : (B, B^+) \to (A, A^+)$ such that $f(\mathrm{Spa}(A, A^+)) \subseteq \mathrm{Spa}(B, B^+)$ consists of vertical generizations of x, there is a unique factorization $f^* = g^* \circ \iota_x^*$.

5. Vertical generizations and residue field maps have the following compatibility. Fix $x \in \mathrm{Spa}(B, B^+)$ with residue field $(K_x, {K_x}^+)$, and consider K_x° the subring of power-bounded elements. Given y a continuous vertical generizations of x we can associate a valuation subring K_y^+ by letting $K_y^+ = \{b \in K_x \mid |b|_y \leq 1\}$. This association gives a bijection between the set of continuous vertical generizations of x and valuation subrings of K_x° containing K_x^+. Moreover, in this case the residue field at y is (K_x, K_y^+).

6. We say that a valuation x is *trivial* if it is equivalent to some valuation for which $\Gamma_x = \{1\}$. The residue field of a trivial valuation is discrete.

7. We say that a valuation is *microbial* if it has a non-trivial rank 1 vertical generization.

8. For technical reasons that will become clear to the reader below, we take the convention of considering trivial valuations as rank 1 valuations.

9. Given a valuation $|\cdot|_x$ of B we define the *characteristic subgroup* of $|\cdot|_x$, denoted by $c\Gamma_x$, as the smallest convex subgroup of Γ_x containing all elements of the form $\gamma = |b|_x$ for $b \in B$ with $1 \leq \gamma$.

10. Given an equivalence class of valuations $|\cdot|_x$ and a convex subgroup $H \subseteq \Gamma_x$ containing $c\Gamma_x$, we define a second equivalence class of valuations $|\cdot|_y$ with $|\cdot|_y : B \to H \cup \{0\}$. We let $|b|_y = |b|_x$ if $|b|_x \in H$ and we let $|b|_y = 0$ otherwise. Any equivalence class of valuations constructed in this way is continuous if $|\cdot|_x$ is continuous. Equivalence classes of valuations constructed in this way are called *horizontal specialization* of x.

11. Horizontal specializations and residue field maps have the following compatibility. Fix $x \in \mathrm{Spa}(B, B^+)$ with residue field $(K_x, {K_x}^+)$. We let K_B be the smallest valuation subring of K_x containing K_x^+ and the image of B in K_x. We get a natural map of Huber pairs $(B, B^+) \to (K_B, K_x^+)$, we consider the induced map $f : \mathrm{Spec}(K_B) \to \mathrm{Spec}(B)$. Horizontal specializations of x are in bijection with prime ideals of B that are in the image of f. Given a convex subgroup H containing $c\Gamma_x$ we can describe the prime ideal $\mathfrak{p}_y$ associated to y as the set of elements of B with $|b|_x < \gamma$ for all $\gamma \in H$. We will denote $|\cdot|_y$ as $|\cdot|_x/\mathfrak{p}_y$.

12. Given a topological space T we construct a partial order on elements of T by letting $t_1 \leq t_2$ whenever $t_1 \in \overline{\{t_2\}}$. We call this partial order the *generization pattern* of T.

13. Vertical generizations and horizontal specializations completely describe the generization pattern of $\mathrm{Spa}(B, B^+)$. More precisely, if y is a vertical generization of x then $x \in \overline{\{y\}}$ and we let xRy. If z is a horizontal specialization of x then $z \in \overline{\{x\}}$ and we let zRx. The generization pattern of $\mathrm{Spa}(B, B^+)$ is the transitive closure of the relation R.

1.2.2 Definitions and basic properties

Definition 1.2.1. *1. We let* $\mathrm{Spo}(B, B^+)$, *denote the subset of* $\mathrm{Spa}(B, B^+) \times \mathrm{Spa}(B, B^+)$ *consisting of pairs* $x = (|\cdot|_x^h, |\cdot|_x^a)$ *such that* $|\cdot|_x^a$ *is a rank* 1 *valuation and a vertical generization of* $|\cdot|_x^h$.

2. Given two elements $b_1, b_2 \in B$ *we let* $U_{b_1 \leq b_2 \neq 0}$ *be the set*

$$\{x \in \mathrm{Spo}(B, B^+) \mid |b_1|_x^h \leq |b_2|_x^h \neq 0\},$$

we call such subsets classical localizations.

3. Given two elements $b_1, b_2 \in B$ *we let* $N_{b_1 << b_2}$ *to be the set*

$$\{x \in \mathrm{Spo}(B, B^+) \mid |b_1|_x^a < |b_2|_x^a \neq 0\},$$

we call such subsets analytic localizations.

4. *We give* $\mathrm{Spo}(B, B^+)$ *the topology generated by classical and analytic localizations, and we call the resulting topological space the olivine spectrum of* (B, B^+).

We will denote by $h : \mathrm{Spo}(B, B^+) \to \mathrm{Spa}(B, B^+)$ the projection onto the first coordinate. This map is continuous and both $\mathrm{Spo}(-, -^+)$ and h are functorial in the category of Huber pairs.

Definition 1.2.2. *Let* $x \in \mathrm{Spo}(B, B^+)$.

1. *We say that* x *is non-analytic if* $|\cdot|^a_x$ *is trivial. We say that a non-analytic point is microbial if* $h(x)$ *is microbial. We say that a non-analytic point is algebraic if* $|\cdot|^h_x$ *is trivial.*

2. *We say that* x *is d-analytic if* $|\cdot|^a_x$ *is non-trivial. Suppose that* x *is d-analytic, we say that it is analytic if* $h(x)$ *is analytic and we say it is meromorphic otherwise.*

3. *We say that* x *is bounded if* $|B|^a_x \leq 1$.

4. *We say that* x *is formal if it is bounded and d-analytic.*

Notice that for any point $x \in \mathrm{Spo}(B, B^+)$ the set $h^{-1}(h(x))$ has at most one d-analytic point and at most one non-analytic point. The cardinality of $h^{-1}(h(x))$ is either one or two. If x is d-analytic then $h^{-1}(h(x))$ has cardinality one only when x is analytic. We warn the reader that although the definitions are designed so that $x \in \mathrm{Spo}(B, B^+)$ is analytic if and only if $h(x) \in \mathrm{Spa}(B, B^+)$ is analytic this is not the case for non-analytic points. Indeed, if x is meromorphic it is d-analytic but $h(x)$ is non-analytic. Meromorphic points behave as analytic points but they are not fully detached from their algebraic nature.

Definition 1.2.3. *We define the support ideal,* $supp(x) \subseteq B$, *as* $supp(h(x))$. *We define the specialization ideal,* $sp(x) \subseteq B^+$ *as the set of elements of* B^+ *with* $|b|^h_x < 1$. *Whenever* x *is bounded we define the deformation ideal, denoted* $def(x)$, *as the prime ideal of elements of* B *for which* $|b|^a_x < 1$.

Notice that x is bounded if and only if $c\Gamma_{x^a} = \{1\}$. Moreover, this only happens if x is either non-analytic or formal. When x is bounded it is non-analytic when $supp(x) = def(x)$ and formal otherwise. We can define the bounded locus, denoted $\mathrm{Spo}(B, B^+)^\dagger$, to be the subset of points that are bounded. This is a closed subset since it is the complement of $\cup_{b \in B} N_{1 << b}$.

Definition 1.2.4. *Let* x *and* y *be two points in* $\mathrm{Spo}(B, B^+)$.

1. *We say that* y *is a vertical generization of* x *(*x *is a vertical specialization respectively) if* $|\cdot|^a_x = |\cdot|^a_y$ *and* $|\cdot|^h_y$ *is a vertical generization of* $|\cdot|^h_x$ *in* $\mathrm{Spa}(B, B^+)$.

2. *We say that* y *is a meromorphic generization of* x *(meromorphic specialization respectively) if* y *is meromorphic,* x *is non-analytic and* $h(x) = h(y)$.

3. We say that y is a formal generization of x (formal specialization respectively) if y is formal, x is non-analytic $def(y) = supp(x)$ and $|\cdot|_x^h = |\cdot|_y^h/def(y)$.

Given $x \in \mathrm{Spo}(B, B^+)$ let $\mathcal{I}^{\leq}(x)$ denote the set of generizations of x in $\mathrm{Spo}(B, B^+)$ and let $\mathcal{I}^{\leq}_{ver}(x)$ denote the set of vertical generizations of x. If the context is clear, for a point $y \in \mathrm{Spa}(B, B^+)$ we will also use $\mathcal{I}^{\leq}_{ver}(y)$ to denote the vertical generizations of y in $\mathrm{Spa}(B, B^+)$. Let us make some easy observations and set some convenient notation:

1. If x is non-analytic it has a meromorphic generization (necessarily unique) if and only if x is a microbial. We denote this generization by x^{mer}.

2. If x is meromorphic it has a unique meromorphic specialization, we denote it by x_{mer}.

3. If x is formal it has a unique formal specialization, we denote it by x_{for}. If x is non-analytic, we let x^{For} denote the set of formal generizations of x.

Example 1.2.5. *If $B = \mathbb{F}_p[[t]]$, the ring of formal power series over $\mathbb{F}_p$ endowed with the discrete topology, then $\mathrm{Spa}(B, B)$ consists of 3 points:*

$$\left\{\eta = |\cdot|_\eta,\ s = |\cdot|_s,\ t = |\cdot|_t\right\}$$

Here $|\cdot|_\eta$ is the trivial valuation with residue field $\mathbb{F}_p((t))$, $|\cdot|_s$ is the trivial valuation with residue field $\mathbb{F}_p$ and $|\cdot|_t$ is the (t)-adic valuation on $\mathbb{F}_p[[t]]$ with residue affinoid field $(\mathbb{F}_p((t)), \mathbb{F}_p[[t]])$. All three valuations have rank 1. The only non-trivial vertical generization in $\mathrm{Spa}(B, B)$ goes from $|\cdot|_t$ to $|\cdot|_\eta$.

On the other hand $Spo(B)$ has 4 points:

$$\left\{\eta = (|\cdot|_\eta, |\cdot|_\eta),\ s = (|\cdot|_s, |\cdot|_s),\ t^a = (|\cdot|_t, |\cdot|_t),\ t^h = (|\cdot|_t, |\cdot|_\eta)\right\}$$

One can verify directly from the definition that $\{\eta\} = U_{1\leq t\neq 0}$, $\{\eta, t^h, t^a\} = U_{0\leq t\neq 0}$, $\{t^a\} = N_{t^2<<t}$ and $\{t^a, s\} = N_{t<<1}$, and that these are the only proper open subsets.

In this example s, η and t^h are non-analytic. Moreover, t^h is microbial, and t^a is both a meromorphic and formal d-analytic point. The generization pattern is as follows: η is a vertical generization of t^h, t^h is the meromorphic specialization of t^a, and s is the formal specialization of t^a. We have that $\mathrm{Spo}(\mathbb{F}_p[[t]], \mathbb{F}_p[[t]])^\dagger = \mathrm{Spo}(\mathbb{F}_p[[t]], \mathbb{F}_p[[t]])$.

The following proposition shows that vertical generizations, formal specializations and meromorphic specializations completely describe the generization pattern in $\mathrm{Spo}(B, B^+)$.

Proposition 1.2.6. *Let $x \in \mathrm{Spo}(B, B^+)$.*

1. If x is d-analytic then $\mathcal{I}^{\leq}(x) = \mathcal{I}^{\leq}_{ver}(x)$.

2. If x is non-analytic then $\mathcal{I}^{\leq}(x) = \mathcal{I}^{\leq}_{ver}(x) \cup \mathcal{I}^{\leq}_{ver}(x^{mer}) \cup (\bigcup_{z\in\mathcal{I}^{\leq}_{ver}(x)} z^{For})$.

Proof. We start by proving the right to left inclusion. Let $y \in \mathcal{I}^{\leq}_{ver}(x) \cup \mathcal{I}^{\leq}_{ver}(x^{mer}) \cup (\bigcup_{z \in \mathcal{I}^{\leq}_{ver}(x)} z^{For})$ if x is non-analytic and let $y \in \mathcal{I}^{\leq}_{ver}(x)$ otherwise. Since h is continuous and $h(y)$ is a generization of $h(x)$ in $\mathrm{Spa}(B, B^+)$ we have that y is contained in every classical localization containing x, so it is enough to check on analytic localizations. Suppose that $x \in N_{b_1 << b_2}$, if y is a vertical generization we have that $|\cdot|^a_y = |\cdot|^a_x$ so $y \in N_{b_1<<b_2}$. If x is non-analytic then $|b_1|^a_x = 0$ and $|b_2|^a_x = 1$, this implies that $|b_1|^a_{x^{mer}} = 0$ and that $|b_2|^a_{x^{mer}} \neq 0$, so $x^{mer} \in N_{b_1<<b_2}$ whenever x^{mer} exists. Moreover, for $y \in x^{For}$, we have that $def(y) = supp(x)$ which gives $|b_1|^a_y < 1$, $|b_2|^a_y = 1$, and $x^{For} \in N_{b_1<<b_2}$.

Now we prove the left to right inclusion, for this take $y \in \mathcal{I}^{\leq}(x)$. Using classical localizations one can deduce that $supp(y) \subseteq supp(x)$, and if x is d-analytic we claim that $supp(y) = supp(x)$. Indeed, let $b \in B$ such that $|b|^a_x \notin \{0, 1\}$, and let $b_1 \in supp(x)$. If $|b|^a_x < 1$ then $|b|^a_y < 1$, which implies that y is d-analytic. Additionally, the inequalities $|b_1|^a_y < |b^n|^a_y$ must hold for all n since $x \in N_{b_1<<b^n}$ and $y \in \mathcal{I}^{\leq}(x)$. In a similar way, if $1 < |b|^a_x$ then $1 < |b|^a_y$ and we may look at the inequalities $|b_1 \cdot b^n|^a_y < |b|^a_y$ instead. In both cases the Archimedean property of rank 1 valuations imply that $b_1 \in supp(y)$. Since the only generizations of $h(x)$ in $\mathrm{Spa}(B, B^+)$ that have the same support as $h(x)$ are vertical generizations we must have $h(y) \in \mathcal{I}^{\leq}_{ver}(h(x))$. Consequently, $y \in \mathcal{I}^{\leq}_{ver}(x)$ holds in the d-analytic case.

Suppose now that x is non-analytic, if $supp(x) = supp(y)$ then we can reason as above to conclude $y \in \mathcal{I}^{\leq}_{ver}(x) \cup \mathcal{I}^{\leq}_{ver}(x^{mer})$. Let us assume there is $b \in supp(x) \setminus supp(y)$. Since $x \in N_{b<<1}$ we have $0 < |b|^a_y < 1$ and that y is d-analytic. By similar reasoning for all $b_1 \in B$ we have $|b \cdot b_1^n|^a_y < 1$ which implies that y is formal, we also have that $supp(x) \subseteq def(y)$. Moreover, for elements $b_2 \notin supp(x)$ we have $x \in U_{b \leq b_2^n \neq 0}$ for all n, this implies that $|b_2|^a_y = 1$ so $def(y) = supp(x)$. If we let $z = y_{for}$ then $supp(z) = supp(x)$ and one can check from the construction of horizontal specializations that z is also a generization of x. As above, we may conclude that $h(z)$ is a vertical generization of $h(x)$, and since both z and x are non-analytic then z is a vertical generization of x. In other words, $z \in \mathcal{I}^{\leq}_{ver}(x)$ and $y \in z^{For}$. $\square$

The olivine spectrum is compatible with completion and rational localization.

Proposition 1.2.7. *Let $(\hat{B}, \hat{B}^+)$ denote the completion of (B, B^+), the map $\mathrm{Spo}(\hat{B}, \hat{B}^+) \to \mathrm{Spo}(B, B^+)$ is a homeomorphism.*

Proof. The map $\mathrm{Spo}(\hat{B}, \hat{B}^+) \to \mathrm{Spo}(B, B^+)$ is a bijection of sets since $\mathrm{Spa}(\hat{B}, \hat{B}^+) \to \mathrm{Spa}(B, B^+)$ is. Since $\mathrm{Spa}(\hat{B}, \hat{B}^+) \cong \mathrm{Spa}(B, B^+)$ classical localizations of $\mathrm{Spa}(\hat{B}, \hat{B}^+)$ are open in $\mathrm{Spa}(B, B^+)$. It is enough to prove that if $f, g \in \hat{B}$ then $N_{g<<f}$ is open in $\mathrm{Spo}(B, B^+)$. Let $x \in N_{g<<f}$ and let $f_x \in B$ with $|f_x|^h_x = |f|^h_x$. We have that

$$U_{f_x \leq f \neq 0} \cap U_{f \leq f_x \neq 0} \cap N_{g<<f} = U_{f_x \leq f \neq 0} \cap U_{f \leq f_x \neq 0} \cap N_{g<<f_x},$$

so we can reduce to the case in which $f \in B$. Take a ring of definition $B_0 \subseteq B$ and an ideal of definition $I \subseteq B_0$ with $|i|^h_x \leq |f|^h_x$ for a finite set of generators $i \in I$. Let $g_x \in B$ such that $g - g_x \in I^2 \cdot \hat{B}_0$. Then

$$(\bigcap_i U_{i \leq f \neq 0}) \cap N_{g_x<<f} = (\bigcap_i U_{i \leq f \neq 0}) \cap N_{g<<f}$$

which proves that the left hand side is also open in $\mathrm{Spo}(B, B^+)$. Indeed, we have the inequalities $|g|^a_x \leq max(|g_x|^a_x, |g - g_x|^a_x)$, $|g_x|^a_x \leq max(|g|^a_x, |g - g_x|^a_x)$ and $|g - g_x|^a_x < |f|^a_x$ by the construction of g_x. □

Proposition 1.2.8. *Let $s, t_1, \ldots, t_n \in B$ such that the t_i generate an open ideal in B and consider the map of Huber pairs $(B, B^+) \to (R, R^+)$ associated to the rational localization $U(\frac{t_1,\ldots,t_n}{s}) \subseteq \mathrm{Spa}(B, B^+)$. The induced map $\mathrm{Spo}(R, R^+) \to \mathrm{Spo}(B, B^+)$ is a homeomorphism onto $h^{-1}(U(\frac{t_1,\ldots,t_n}{s}))$.*

Proof. The only thing to verify is that for every $r_1, r_2 \in R$ the analytic localization $N_{r_1<<r_2} \subseteq \mathrm{Spo}(R, R^+)$ is also open in $\mathrm{Spo}(B, B^+)$. By proposition 1.2.7 and the construction of R as a rational localization we may assume that $r_1, r_2 \in B[\frac{1}{s}] \subseteq R$ since R is defined as the completion of $B[\frac{1}{s}]$ under certain linear topology. Let $r_1 = \frac{b_1}{s^{n_1}}$ and $r_2 = \frac{b_2}{s^{n_2}}$ and let $m = n_1 - n_2$. Then $N_{r_1<<r_2} = N_{b_1<<b_2 \cdot s^m} \cap \mathrm{Spo}(R, R^+)$ when m is non-negative and $N_{r_1<<r_2} = N_{b_1 \cdot s^m<<b_2} \cap \mathrm{Spo}(R, R^+)$ otherwise. □

The following example generalizes example 1.2.5. We encourage the reader to understand this example carefully before moving on. This will also be used in the proof of lemma 1.2.33 and theorem 4.

Example 1.2.9. *Suppose that B and B^+ are valuation rings with the same fraction field and suppose that B has the discrete topology. To describe $\mathrm{Spo}(B, B^+)$ it is enough to realize that $\mathrm{Spo}(B, B^+) \subseteq \mathrm{Spo}(B^+, B^+)$ and that it acquires the subspace topology. Indeed, it corresponds to the intersection of the classical localizations $\cap_{b \in B \setminus B^+} U_{0 \leq b \neq 0} \subseteq \mathrm{Spo}(B, B^+)$ (often times only one of the terms in this intersection is needed). To simplify notation we only describe explicitly $\mathrm{Spo}(B^+, B^+)$, the advantage of this case is that all points are bounded and the deformation ideal def makes sense.*

Consider the map $\mathrm{Spo}(B^+, B^+) \to \mathrm{Spec}(B^+)^3$ given by

$$q \mapsto (supp(q), def(q), sp(q)).$$

The map is injective and the image consists of the set of triples (q_1, q_2, q_3) such that $q_1 \subseteq q_2 \subseteq q_3$, and such that the closed interval $[q_1, q_2]$ (in the sense of ordered sets) has cardinality one or two. A triple $q = (q_1, q_2, q_3)$ is d-analytic (necessarily meromorphic) if and only if $q_1 \neq q_2$, in this case there exists an element $b \in B^+$ with $b \in q_2 \setminus q_1$. We have that $q \in \mathcal{I}^{\leq}_{ver}(r)$ if $q_1 = r_1$, $q_2 = r_2$ and $q_3 \subseteq r_3$. Also, $q = r^{mer}$ if $r_1 = q_1 = r_2$, $q_2 \neq q_1$ and $r_3 = q_3$. With this setup formal generizations are unique and r is the formal specialization of q (i.e. $r = q_{for}$) if $q_1 \neq q_2$, $r_1 = q_2 = r_2$ and $r_3 = q_3$.

To each element $0 \neq g \in B^+ \setminus (B^+)^\times$ we can associate two ideals q^+_g and q^-_g where q^+_g is the largest prime ideal not containing g and q^-_g is the smallest prime ideal containing g. We have that $q^+_g \subseteq q^-_g$ and that the interval $[q^+_g, q^-_g]$ has cardinality two.

When $\frac{f}{g} \in B^+$ then $U_{f \leq g \neq 0} = U_{0 \leq g \neq 0}$ and consists of triples (q_1, q_2, q_3) with $q_1 \leq q^-_g$. On the other hand if $\frac{g}{f} \in B^+$ we can let $b = \frac{g}{f}$ then $U_{f \leq g \neq 0} = U_{1 \leq b \neq 0}$ and it consists of triples with $q_3 \leq q^-_b$. Notice that the two families of open sets $U_{1 \leq b \neq 0}$ and $U_{0 \leq g \neq 0}$ are nested (i.e.

one of $U_{1\leq b_2\neq 0} \subseteq U_{1\leq b_1\neq 0}$ or $U_{1\leq b_1\neq 0} \subseteq U_{1\leq b_2\neq 0}$ holds). In particular, finite intersections of classical localizations are all of the form $U_{1\leq b\neq 0} \cap U_{0\leq g\neq 0}$ for some $b, g \in B^+$.

When $n = \frac{f}{g} \in B^+$ then $N_{g<<f}$ is empty and $N_{f<<g} = U_{0\leq g\neq 0} \cap N_{n<<1}$. The set $N_{n<<1}$ consists of the triples $q = (q_1, q_2, q_3)$ such that $q_n^+ \leq q_2$. The family of sets $N_{n<<1}$ is again nested.

In summary, if $x \in U \subseteq \mathrm{Spo}(B^+, B^+)$ for U an open subset there are elements $g, b, n \in B^+$ with $x \in U_{0\leq g\neq 0} \cap U_{1\leq b\neq 0} \cap N_{n<<1} \subseteq U$ and elements of $U_{0\leq g\neq 0} \cap U_{1\leq b\neq 0} \cap N_{n<<1} \subseteq U$ correspond to triples (q_1, q_2, q_3) satisfying: $q_n^+, q_1 \subseteq q_2 \subseteq q_3 \subseteq q_b^-$ and $q_1 \subseteq q_g^-$.

1.2.3 Olivine Huber pairs

For the rest of the section (B, B^+) will denote a complete Huber pair over $\mathbb{Z}_p$.

Proposition 1.2.10. *If R is a Tate Huber pair the projection map $h : \mathrm{Spo}(R, R^+) \to \mathrm{Spa}(R, R^+)$ is a homeomorphism.*

Proof. Since (R, R^+) is Tate there are no trivial continuous valuations in $\mathrm{Spa}(R, R^+)$. In particular every point in $\mathrm{Spo}(R, R^+)$ is d-analytic and h is injective. If x^a is the maximal generization of x in $\mathrm{Spa}(R, R^+)$ then $h^{-1}(x) = \{(x, x^a)\}$. It is enough to prove that $h(N_{r_1<<r_2})$ is open. If $\varpi \in R$ is a topologically nilpotent unit, then

$$h(N_{r_1<<r_2}) = \bigcup_{0<n} \{z \in \mathrm{Spa}(R, R^+) \mid |r_1^n|_z \leq |r_2^n \varpi|_z \neq 0\}$$

Indeed, a point $x \in \mathrm{Spa}(R, R^+)$ is in $h(N_{r_1<<r_2})$ if $|r_1|_{x^a} < |r_2|_{x^a}$. By the Archimedean property of rank 1 valuations there is $n \in \mathbb{N}$ such that $(\frac{|r_1|_{x^a}}{|r_2|_{x^a}})^n \leq |\varpi|_{x^a}$ since ϖ is a unit and $|\varpi|_{x^a} > 0$. On the other hand, if $|r_1^n|_x \leq |r_2^n \cdot \varpi|_x$ we also have $|r_1^n|_{x^a} \leq |r_2^n \cdot \varpi|_{x^a} < |r_2^n|_{x^a}$ since ϖ is topologically nilpotent. $\square$

If $m : (B, B^+) \to (R, R^+)$ is a map of Huber pairs, we denote by $\mathrm{Spo}(m) : \mathrm{Spo}(R, R^+) \to \mathrm{Spo}(B, B^+)$ the corresponding map of olivine spectra. In case (R, R^+) is Tate we have a continuous map $\mathrm{Spo}(m) \circ h^{-1} : \mathrm{Spa}(R, R^+) \to \mathrm{Spo}(B, B^+)$. When the context is clear, we also abbreviate $\mathrm{Spo}(m) \circ h^{-1}$ by $\mathrm{Spo}(m)$.

Remark 1.2.11. *The topological considerations in what follows can be done purely in the context of adic spaces without any reference to perfectoid spaces. To do this one substitutes $|\mathrm{Spd}(B, B^+)|$ by $\mathrm{Spo}(B, B^+)'$ where this second space has $\mathrm{Spo}(B, B^+)$ as underlying set but has the strongest topology making $\mathrm{Spo}(m)$ continuous for maps $m : (B, B^+) \to (R, R^+)$ ranging over all Tate Huber pairs. We do not pursue this.*

We define a map $\pi : |\mathrm{Spd}(B, B^+)| \to \mathrm{Spo}(B, B^+)$ as follows. Let $[x] \in |\mathrm{Spd}(B, B^+)|$ be represented by a geometric point $x : \mathrm{Spa}(C_x, {C_x}^+) \to \mathrm{Spd}(B, B^+)$, and let $s \in \mathrm{Spa}(C_x, {C_x}^+)$ denote the unique closed point. Recall that the map is given by an untilt $C_x^\sharp$ and a map of Huber pairs $f_x^* : (B, B^+) \to (C_x^\sharp, {C_x^\sharp}^+)$. We define $\pi([x])$ to be $\mathrm{Spo}(f_x)(\flat^{-1}(s))$ where $\flat : \mathrm{Spa}(C_x^\sharp, {C_x^\sharp}^+) \to \mathrm{Spa}(C_x, {C_x}^+)$ is the tilting homeomorphism. This map doesn't depend of the representative picked.

Proposition 1.2.12. *The map $\pi : |\mathrm{Spd}(B, B^+)| \to \mathrm{Spo}(B, B^+)$ defined above is continuous and bijective.*

Proof. To prove π is continuous one has to show that for any map $f : \mathrm{Spa}(R, R^+) \to \mathrm{Spd}(B, B^+)$ with (R, R^+) perfectoid, the composition

$$|\mathrm{Spa}(R, R^+)| \to |\mathrm{Spd}(B, B^+)| \to \mathrm{Spo}(B, B^+)$$

is continuous. But if $f^\sharp : (B, B^+) \to (R^\sharp, R^{\sharp+})$ is the map of Huber pairs associated to f then $\pi \circ |f|$ is given by $\mathrm{Spo}(f^\sharp) \circ \flat^{-1}$. Let us prove injectivity, take two geometric points $y_i : \mathrm{Spa}(C_i, {C_i}^+) \to \mathrm{Spd}(B, B^+)$ and suppose that $\pi(y_1) = \pi(y_2)$. We need to show that $[y_1] = [y_2]$. Let x be the common image in $\mathrm{Spo}(B, B^+)$ and let $(K_{h(x)}, {K_{h(x)}}^+)$ be the affinoid residue field of $h(x)$ in $\mathrm{Spa}(B, B^+)$. The map $(B, B^+) \to (C_i^\sharp, {C_i^\sharp}^+)$ factor through $(B, B^+) \to (K_{h(x)}, {K_{h(x)}}^+)$. We split our analysis in three cases.

Case 1: Suppose that x is analytic. In this case the closed point s_i of $\mathrm{Spa}(C_i^\sharp, {C_i^\sharp}^+)$ maps to $h(\pi(x))$ which is in the analytic locus $\mathrm{Spa}(B, B^+)^{an}$. This case follows from the bijectivity of $|X^\diamondsuit| \to |X|$ for analytic pre-adic spaces (See remark 1.1.31).

Case 2: Suppose that x is meromorphic, we have that $h(x)$ is non-analytic in $\mathrm{Spa}(B, B^+)$. Let $K^\circ_{h(x)} := \{k \in K_{h(x)} \mid |k|^a_x \leq 1\}$ since $|\cdot|^a_x$ is non-trivial $K^\circ_{h(x)}$ is a proper valuation subring of $K_{h(x)}$. Choose $b \in B$ such that either $0 < |b|^a_x < 1$ or $|b|^a_x > 1$, then the subspace topology on $(K^\circ_{h(x)}) \subseteq_{y_i^*} O_{C_i^\sharp}$ coincides with the (b)-adic topology or, respectively, the $(\frac{1}{b})$-adic topology. Taking the completion with respect to this topology we get a commutative diagram:

$$\begin{tikzcd}
& \mathrm{Spa}(C_1, {C_1}^+) \arrow[d, "p_1'"] \arrow[dr, bend left, "y_1"] & \\
\mathrm{Spa}(C_2, {C_2}^+) \arrow[r, "p_2'"] \arrow[dr, bend right, "y_2"'] & \mathrm{Spd}(\hat{K}_{h(x)}, \hat{K}^+_{h(x)}) \arrow[d, "\iota_x"] & \\
& \mathrm{Spd}(K_{h(x)}, {K_{h(x)}}^+) &
\end{tikzcd}$$

The maps p_i', map the respective closed points to the same underlying topological point of $\mathrm{Spd}(\hat{K}_{h(x)}, \hat{K}^+_{h(x)})$. Since $\mathrm{Spa}(\hat{K}_{h(x)}, \hat{K}^+_{h(x)})$ is analytic we can conclude as in the first case.

Case 3: Suppose that x is non-analytic, in this case $h(x)$ is non-analytic in $\mathrm{Spa}(B, B^+)$. We have that $(K_{h(x)}, {K_{h(x)}}^+)$ is given the discrete topology. Since $|\cdot|^a_x$ is trivial we have that $y_i(K_{h(x)}) \subseteq O^{\sharp,\times}_{C_i}$. After choosing pseudo-uniformizers $\varpi_i \in O_{C_i^\sharp}$ we may extend the y_i to continuous adic maps of topological rings $p_i'^* : K_{h(x)}[[t]] \to O_{C_i^\sharp}$ where $K_{h(x)}[[t]]$ is given the (t)-adic topology. These induce the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Spa}(C_1, {C_1}^+) \arrow[d, "p'_1"] \arrow[ddr, bend left, "y_1"] & \\
\mathrm{Spa}(C_2, {C_2}^+) \arrow[r, "p'_2"] \arrow[drr, bend right, "y_2"] & \mathrm{Spd}(K_{h(x)}((t)), K^+_{h(x)} + t \cdot K_{h(x)}[[t]]) \arrow[dr, "\iota_x"] & \\
 & & \mathrm{Spd}(K_{h(x)}, {K_{h(x)}}^+)
\end{tikzcd}$$

The maps p'_i again send the closed point to the same point in $\mathrm{Spd}(K_{h(x)}((t)), K^+_{h(x)} + t \cdot K_{h(x)}[[t]])$ and this space is again the diamond associated to an analytic space so we may conclude as above. This finishes the proof of injectivity. The argument given above also explains how to construct a geometric point of $p_x : \mathrm{Spa}(C, C^+) \to \mathrm{Spa}(B, B^+)$ with $\mathrm{Spo}(p_x)(s) = x$. Indeed, we can take a completed algebraic closure of $K_{h(x)}$ ($\hat{K}_{h(x)}$, or $K_{h(x)}((t))$ respectively) when x is analytic (meromorphic or non-analytic respectively). $\square$

Definition 1.2.13. *Whenever x is d-analytic we let $(K_x, {K_x}^+)$ denote $(\hat{K}_{h(x)}, \hat{K}^+_{h(x)})$, and if x is non-analytic we let $(K_x, {K_x}^+)$ denote $(K_{h(x)}((t)), K^+_{h(x)} + t \cdot K_{h(x)}[[t]])$ as in the proof of proposition 1.2.12. In both cases we call $(K_x, {K_x}^+)$ the pseudo-residue field at x.*

Remark 1.2.14. *The pseudo-residue field map $\mathrm{Spo}(K_x, {K_x}^+) \to \mathrm{Spo}(B, B^+)$ is a homeomorphism onto its image. The functor $\mathrm{Spd}(K_x, {K_x}^+) \to \mathrm{Spd}(B, B^+)$ surjects onto the subsheaf of $\mathrm{Spd}(B, B^+)$ consisting of maps that factor through $\mathcal{I}^{\leq}_{ver}(x)$, but when x is non-analytic the map $\mathrm{Spd}(K_x, {K_x}^+) \to \mathrm{Spd}(B, B^+)$ is not injective. Actually, when x is non-analytic and $|\cdot|^h_x$ is non-trivial the subsheaf of points that factor through $\mathcal{I}^{\leq}_{ver}(x)$ is not representable by an adic space.*

Corollary 1.2.15. *For any map of Huber pairs $m^* : (B_1, {B_1}^+) \to (B_2, {B_2}^+)$ the map $\mathrm{Spo}(m)$ is compatible with vertical generization. More precisely, if $x \in \mathrm{Spo}(B_2, {B_2}^+)$, $y = \mathrm{Spo}(m)(x)$ and y' is a vertical generization of y then there exist x', a vertical generization of x, with $\mathrm{Spo}(m)(x') = y'$.*

Proof. Given $x \in \mathrm{Spo}(B_2, {B_2}^+)$ and $y \in \mathrm{Spo}(B_1, {B_1}^+)$ as in the statement we may, after making some choices if necessary, construct the following commutative diagram of pseudo-residue fields:

$$\begin{tikzcd}
\mathrm{Spd}(K_x, {K_x}^+) \arrow[r] \arrow[d] & \mathrm{Spd}(K_y, {K_y}^+) \arrow[d] \\
\mathrm{Spd}(B_2, {B_2}^+) \arrow[r] & \mathrm{Spd}(B_1, {B_1}^+)
\end{tikzcd}$$

Since the map $\mathrm{Spd}(K_x, {K_x}^+) \to \mathrm{Spd}(K_y, {K_y}^+)$ is a map of locally spatial diamonds it is generalizing and consequently surjective. But $|\mathrm{Spd}(K_x, {K_x}^+)| = \mathcal{I}^{\leq}_{ver}(x)$ and analogously for y. $\square$

Lemma 1.2.16. *The topological spaces $\mathrm{Spo}(B, B^+)$ and $|\mathrm{Spd}(B, B^+)|$ have the same generization pattern.*

Proof. Since the map $|\mathrm{Spd}(B, B^+)| \to \mathrm{Spo}(B, B^+)$ is continuous the generization pattern of $|\mathrm{Spd}(B, B^+)|$ is smaller than that of $\mathrm{Spo}(B, B^+)$, it is enough by proposition 1.2.6 to prove that formal, meromorphic and vertical specializations are specializations in $|\mathrm{Spd}(B, B^+)|$. For any $x \in \mathrm{Spo}(B, B^+)$ the pseudo-residue field map $\iota_x : \mathrm{Spd}(K_x, {K_x}^+) \to \mathrm{Spd}(B, B^+)$ in the proof of proposition 1.2.12 defines a bijection onto $\mathcal{I}^{\leq}_{ver}(x)$ which proves that vertical specializations are specializations in $\mathrm{Spd}(B, B^+)$.

Let $x \in \mathrm{Spo}(B, B^+)$ be d-analytic and let b such that $|b|^a_x \notin \{0, 1\}$. Let $p : \mathrm{Spa}(C, C^+) \to \mathrm{Spa}(B, B^+)$ be a geometric point mapping to x and let $\varpi \in C^{\circ\circ}$ be either $p^*(b)$ or $\frac{1}{p^*(b)}$. To this choice we will associate two product of points as follows. Let $R^+ = \prod_{i=1}^\infty C^+$, let $\varpi_0 = (\varpi^{\frac{1}{n}})_{n=1}^\infty$ and $\varpi_\infty = (\varpi^n)_{n=1}^\infty$. Let R_0^+ (R_∞^+ respectively) be R^+ endowed with the ϖ_0-topology (ϖ_∞-topology respectively), and let $R_0 = R_0^+[\frac{1}{\varpi_0}]$ ($R_\infty = R_\infty^+[\frac{1}{\varpi_\infty}]$ respectively). We have diagonal maps of rings $C^+ \to R_\infty^+$ and $C \to R_\infty$, but we warn the reader that this maps are not continuous. On the other hand, the map $C^+ \to R_0^+$ is continuous but ϖ is not invertible in R_0 so the map does not extend to a map $C \to R_0$. Intuitively speaking, the product of points $\mathrm{Spa}(R_\infty, {R_\infty}^+)$ "converges outside" of the locus in which ϖ is topologically nilpotent and the product of points $\mathrm{Spa}(R_0, {R_0}^+)$ "converges outside" of the locus in which ϖ is invertible.

Suppose that x is meromorphic, then the affinoid residue field $(K_{h(x)}, {K_{h(x)}}^+)$ is given the discrete topology. In particular, the diagonal map $f : B \to K_{h(x)} \to R_\infty$ is continuous and defines a map $\mathrm{Spa}(R_\infty, {R_\infty}^+) \to \mathrm{Spa}(B, B^+)$. The space of connected components $\pi_0(|\mathrm{Spa}(R_\infty, {R_\infty}^+)|)$ is the Stone-Čech compactification of $\mathbb{N}$ which has as underlying set the set of ultrafilters of $\mathbb{N}$. Principal ultrafilters $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ define inclusions $\iota_n : \mathrm{Spa}(C, C^+) \to \mathrm{Spa}(R_\infty, {R_\infty}^+)$ that correspond to the nth-projection in the coordinate rings. In particular, the closed point of a principal connected component maps to x under $\mathrm{Spo}(f)$. We claim that the closed point of a non-principal connected component maps to x_{mer}. It is enough to construct a commutative diagram as below:

$$\begin{tikzcd}
\mathrm{Spa}(C_\mathcal{U}, {C_\mathcal{U}}^+) \arrow[r] \arrow[d] & \mathrm{Spa}(K_{x_{mer}}, {K_{x_{mer}}}^+) \arrow[d] & \\
\mathrm{Spa}(R_\infty, {R_\infty}^+) \arrow[r] & \mathrm{Spa}(K_{h(x)}, {K_{h(x)}}^+) \arrow[r] & \mathrm{Spa}(B, B^+)
\end{tikzcd}$$

We claim that the natural map $K_{h(x)} \to C_\mathcal{U}$ maps to $O_{C_\mathcal{U}}$. Indeed, it is enough to prove $\varpi_\infty \cdot K_{h(x)} \subseteq O_{C_\mathcal{U}}$ since then every element of $K_{h(x)}$ would be power bounded. Clearly $K^+_{h(x)} \subseteq O_{C_\mathcal{U}}$ and since $K_{h(x)} = K^+_{h(x)}[b, \frac{1}{b}]$ it is enough to prove that $\frac{\varpi_\infty}{\varpi^n} \in O_{C_\mathcal{U}}$ for $n \in \mathbb{N}$. Clearly $\frac{\varpi_\infty}{\varpi^n} \in \prod_{i=n+1}^\infty O_C$ and since our ultrafilter is non-principal complements of finite sets are in $\mathcal{U}$, which finishes the proof of the claim.

By letting t map to ϖ_∞ we get a map $K_{h(x)}((t)) \to C_\mathcal{U}$, the intersection of $K_{h(x)}[[t]]$ with $C^+_\mathcal{U}$ in $O_{C_\mathcal{U}}$ is $K^+_{h(x)} + t\cdot K_{h(x)}[[t]] = K^+_{x_{mer}}$ which gives our factorization. Since the set of points that are contained in a principal connected component and that are closed in $|\mathrm{Spa}(R, R^+)|$ is dense within the set of closed points of $|\mathrm{Spa}(R, R^+)|$, meromorphic specializations in $\mathrm{Spo}(B, B^+)$ are specializations in $|\mathrm{Spd}(B, B^+)|$.

Suppose now that x is formal, since $|B|_x^a \leq 1$ the map $(B, B^+) \to (C, C^+)$ factors through a map to (O_C, C^+) and we have $def(x) = B \cap C^{\circ\circ}$. This allows us to define a map $\mathrm{Spa}(R_0, R_0{}^+) \to \mathrm{Spa}(B, B^+)$. As in the previous case the space of connected components of $\mathrm{Spa}(R_0, R_0{}^+)$ is Stone-Čech compactification of $\mathbb{N}$, principal connected components of $\mathrm{Spa}(R_0, R_0{}^+)$ map to x in $\mathrm{Spo}(B, B^+)$ and we will show that the non-principal ones map to x_{for}.

Let $k = O_C/C^{\circ\circ}$ and $k^+ = C^+/C^{\circ\circ}$, it is enough to prove that the map $(O_C, C^+) \to (C_{\mathcal{U}}, C_{\mathcal{U}}{}^+)$ factors as:

$$(O_C, C^+) \to (k, k^+) \to (k\,((t))\,, k^+ + t \cdot k\,[[t]]) \to (C_{\mathcal{U}}, C_{\mathcal{U}}{}^+)$$

Now $\frac{\varpi}{\varpi_0^n} \in \prod_{i=n+1}^{\infty} O_C$ which implies that $|\varpi|_{\mathcal{U}} \leq |\varpi_0^n|_{\mathcal{U}}$. Since ϖ_0 is a pseudo-uniformizer in $C_{\mathcal{U}}$ this implies $|\varpi|_{\mathcal{U}} = 0$. Clearly $k \subseteq O_{C_{\mathcal{U}}}$ and we may send t to ϖ_0 to construct our factorization. We may conclude the proof that formal specializations are specializations in $|\mathrm{Spo}(B, B^+)|$ as in the previous case. $\square$

We say that a Huber pair is formal if it is of the form (B, B) where B is an I-adic ring with finitely generated ideal I. For the moment we restrict to studying the olivine spectrum of formal Huber pairs. The main technical advantage of restricting to this case is that the open unit ball over $\mathrm{Spd}(B, B)$ is easy to describe. Indeed, it is represented by $\mathrm{Spd}(B[[t]], B[[t]])$ when this ring is given the (I, t)-adic topology.

Proposition 1.2.17. *Let (B, B) be a formal Huber pair then the map $|\mathrm{Spd}(B, B)| \to \mathrm{Spo}(B, B)$ is a homeomorphism.*

Proof. By proposition 1.2.12 we the map is a continuous bijection. Let

$$Y = \mathrm{Spa}(B[[t]], B[[t]])^{t \neq 0}$$

and recall that $|Y| = |Y^{\lozenge}|$ since this is an analytic pre-adic space. Let U be an open subset of $|\mathrm{Spd}(B, B^+)|$, let $x \in U$ and let $y \in Y$ be a point mapping to x, such that $|t|_y \neq 0$. We will construct a neighborhood of x contained in U that is open in $\mathrm{Spo}(B, B)$.

Given a classical localization $U_{b_1 \leq b_2 \neq 0}$ or an analytic localization $N_{b_1 << b_2}$ containing x we choose quasi-compact neighborhoods of y in $\mathrm{Spa}(B\,[[t]], B\,[[t]])$, that we denote $U_{b_1, b_2, y}$ and $N_{b_1, b_2, y}$, whose image in $\mathrm{Spo}(B, B)$ are contained in $U_{b_1 \leq b_2 \neq 0}$ and $N_{b_1 << b_2}$ respectively. The construction is as follows, given the classical localization $U_{b_1 \leq b_2 \neq 0}$ we pick a finite set of elements $S \subseteq B$ and a positive integer n such that $|s|_y \leq |b_2|_y$ for $s \in S$, that $|t^n|_y \leq |b_2|_y$, and that the ideal generated by S is open in B. We let $U_{b_1, b_2, y}$ be the rational localization $U(\frac{S, t^n, b_1}{b_2}) \subseteq \mathrm{Spa}(B[[t]], B[[t]])$. Rational localizations of affinoid adic spaces are always quasi-compact open subsets and clearly $\mathrm{Spo}(f)(h^{-1}(U_{b_1, b_2, y})) \subseteq U_{b_1 \leq b_2 \neq 0}$.

Analogously, given $N_{b_1 << b_2}$ we pick a set S and two positive integers, n_1 and n_2, such that $|b_1^{n_1}|_y \leq |b_2^{n_1} \cdot t|_y$, that $|s|_y \leq |b_2^{n_1} \cdot t|_y$ for $s \in S$, that $|t^{n_2}|_y \leq |t \cdot b_2^{n_1}|_y$ and that S generates an open ideal in B. We let $N_{b_1, b_2, y} = U(\frac{S, t^{n_2}, b_1^{n_1}}{b_2^{n_1} \cdot t})$. Since t is topologically nilpotent in $B[[t]]$, for any point $z \in \mathrm{Spa}(B[[t]], B[[t]])$ we must have $|t|_z < 1$, which proves $\mathrm{Spo}(f)(h^{-1}(N_{b_1,b_2,y})) \subseteq N_{b_1<<b_2}$. Notice that $N_{b_1,b_2,y} \subseteq \mathrm{Spa}(B[[t]], B[[t]])^{t\neq 0}$.

Let X denote the intersection of all neighborhoods of y of the form $N_{b_1,b_2,y}$ and $U_{b_1,b_2,y}$ that were chosen in this way, then $\mathrm{Spo}(f)(X)$ is contained in $\mathcal{I}^{\leq}(x)$. By lemma 1.2.16 $|\mathrm{Spd}(B, B^+)|$ and $\mathrm{Spo}(B, B)$ have the same generization pattern, so we also have that $\mathrm{Spo}(f)(X) \subseteq U$. We have that $\mathrm{Spo}(f)^{-1}(U)$ is open in $\mathrm{Spa}(B[[t]], B[[t]])^{t\neq 0}$ and that the two families, $U_{b_1,b_2,y} \cap N_{0,1,y}$ and $N_{b_1,b_2,y}$, consist of quasi-compact open subsets of $\mathrm{Spa}(B[[t]], B[[t]])^{t\neq 0}$. A compactness argument in the patch topology of $\mathrm{Spa}(B[[t]], B[[t]])$ will prove that a finite intersection of these neighborhoods is contained in $\mathrm{Spo}(f)^{-1}(U)$. We prove below that the image under $\mathrm{Spo}(f)$ of such a finite intersection is open in $\mathrm{Spo}(B, B)$.

It is enough to show that if a set Z is a finite intersections of sets of the form

$$V_{b'_1,b'_2} := \{z \in \mathrm{Spa}(B[[t]], B[[t]])^{t\neq 0} \mid |b'_1|_z \leq |b'_2|_z \neq 0\}$$

where $b'_1 \in B \cup \{t^n\}_{n\in\mathbb{N}}$ and $b'_2 \in B \cup t \cdot B$, then $\mathrm{Spo}(f)(Z)$ is open in $\mathrm{Spo}(B, B)$. Observe that if $b'_1, b'_2 \in B$ then $V_{b'_1,b'_2} = \mathrm{Spo}(f)^{-1}(U_{b'_1\leq b'_2\neq 0})$ and that for any Z as above we have $\mathrm{Spo}(f)(Z \cap V_{b'_1,b'_2}) = \mathrm{Spo}(f)(Z) \cap U_{b'_1\leq b'_2\neq 0}$. This allow us to reduce to the case in which Z is an intersections of opens such that at least one of $b'_1 \in \{t^n\}_{n\in\mathbb{N}}$ or $b'_2 = b_2 \cdot t$ holds.

Let T^n_Z be the subset of B for which either $V_{t^n,b}$ or $V_{t^{n+1},(b\cdot t)}$ appear in the expression of Z as an intersection, we let $T^{<<}_Z$ be the set of pairs $(b_1, b_2) \in B \times B$ such that $V_{b_1,(b_2\cdot t)}$ appears in the expression of Z as an intersection, and we let T^-_Z and T^+_Z denote the image of $T^{<<}_Z$ under the projection onto the first and second factors respectively. We claim, and prove below, that $\mathrm{Spo}(f)(Z)$ is the intersection of all the sets of the form $U_{b_1^n\leq b_2^n\cdot b_3\neq 0}$ where $(b_1, b_2) \in T^{<<}_Z$ and $b_3 \in T^n_Z$ and all the sets of the form $N_{b_1<<b_2}$, with $(b_1, b_2) \in T^{<<}_Z$. This proves $\mathrm{Spo}(f)(Z)$ is open.

Let $z \in Z$ with associated rank 1 point $z^a \in Z$, let $w = \mathrm{Spo}(f)(z)$ and fix b_1, b_2 and b_3 as above. By raising to the nth-power we have that $|b_1^n|_z \leq |b_2^n|_z \cdot |t^n|_z$ and $|t^n|_z \leq |b_3|_z$ hold. In particular,

$$|b_1^n|_z = |b_1^n|^h_w \leq |b_2^n|^h_w \cdot |b_3|^h_w = |b_2^n|_z \cdot |b_3|_z$$

holds as well and we can conclude that $w \in U_{b_1^n\leq b_2^n\cdot b_3\neq 0}$. Similarly, since t is topologically nilpotent we have $|t|_{z^a} < 1$ which implies that $|b_1|_{z^a} < |b_2|_{z^a}$ and consequently that $|b_1|^a_w < |b_2|^a_w$. This says that $w \in N_{b_1<<b_2}$.

To prove the converse containment given a point w in the intersection of those sets we need to construct a lift landing in Z. Pick a geometric point $q : \mathrm{Spa}(C, C^+) \to \mathrm{Spa}(B, B)$ mapping to w in $\mathrm{Spo}(B, B)$, the choice of an element $\varpi \in C^{\circ\circ,\times}$ defines a lift of q to a map $\mathrm{Spa}(C, C^+) \to \mathrm{Spa}(B[[t]], B[[t]])^{t\neq 0}$ simply by letting t map to ϖ. If w is non-analytic then $|b_1|^a_w = 0$ for every $b_1 \in T^-_Z$ and $|b_2|^a_w = |b_3|^a_w = 1$ for every $b_2 \in T^+_Z$ and $b_3 \in T^n_Z$. In this case, any choice of ϖ defines a lift landing inside of Z.

If w is d-analytic ϖ must be chosen more carefully. Since C is algebraically closed we may choose nth-roots of (b_3) for all $b_3 \in T^n_Z$. For a lift of q to land in Z, ϖ must satisfy the following: $|\varpi|_q \leq |(b_3)^{\frac{1}{n}}|_q$ for all $b_3 \in T^n_Z$ and $\frac{|(b_1)|_q}{|(b_2)|_q} \leq |\varpi|_q$ for all $(b_1, b_2) \in T^{<<}_Z$. We let m be the smallest of the values in Γ_q of the form $|b_3^{\frac{1}{n}}|_q$ with $b_3 \in T^n_Z$ and we let M be the largest of the values of the form $|\frac{b_1}{b_2}|_q$ with $(b_1, b_2) \in T_Z^{<<}$. Since $w \in U_{b_1^n \leq b_2^n \cdot b_3 \neq 0}$ we have $M \leq m$. Since $w \in N_{b_1 << b_2}$ for all pairs $(b_1, b_2) \in T_Z^{<<}$ we also have $M < 1$. Any $\varpi \in C$ with $|\varpi|_q < 1$ and $M \leq |\varpi|_q \leq m$ defines a lift of q in Z. This finishes the proof of the first claim.

$\square$

Definition 1.2.18. *Let (B, B^+) be a complete Huber pair over $\mathbb{Z}_p$, we say that (B, B^+) is olivine if the map $|\mathrm{Spd}(B, B^+)| \to \mathrm{Spo}(B, B^+)$ is a homeomorphism.*

Question 1.2.19. *Is every complete Huber pair over $\mathbb{Z}_p$ an olivine Huber pair?*

We have some partial progress in answering this question. Although we do not know what to expect in full generality, for all of the Huber pairs that we consider this is true. Let us clarify. By remark 1.1.31 we know that Tate Huber pairs are olivine. By proposition 1.2.17 we know that formal Huber pairs are olivine. Proposition 1.2.8 allows us to conclude that if $(B, B^+) \to (R, R^+)$ induces an open immersion $\mathrm{Spa}(R, R^+) \subseteq \mathrm{Spa}(B, B^+)$ and $\mathrm{Spa}(B, B^+)$ is olivine then $\mathrm{Spa}(R, R^+)$ is olivine. Moreover, we can conclude that being olivine is a property that can be verified locally in the analytic topology of $\mathrm{Spa}(B, B^+)$. The following proposition is the most general criterion we could come up with.

Proposition 1.2.20. *Suppose that (B, B^+) is a complete Huber pair over $\mathbb{Z}_p$ such that B^+ is a ring of definition with ideal of definition I and such that B is a finite type B^+-algebra. Then (B, B^+) is an olivine Huber pair.*

Proof. Write $B = B^+[b_1, \ldots, b_n]$ we prove that (B, B^+) is olivine by induction on the size of n, the case $n = 0$ is the content of proposition 1.2.17. Let $\mathrm{Spa}(R, R^+)$ be the rational localization corresponding to $\{x \in \mathrm{Spa}(B, B^+) \mid |b_1|_x \leq |1|_x \neq 0\}$, we claim that (R, R^+) is olivine. By proposition 1.2.7 we may compute R without taking completions. Up to completion R^+ is the integral closure of $B^+[b_1]$ in B and the underlying rings of R and B coincide (although they are not homeomorphic). Clearly R is generated over R^+ by less than n elements so by induction it is olivine. We let $\mathrm{Spa}(S, S^+)$ be the rational localization corresponding to $\{x \in \mathrm{Spa}(B, B^+) \mid |1|_x \leq |b_1|_x \neq 0\}$. Up to completion $S = B[\frac{1}{b_1}]$ as rings and S^+ is the integral closure of $B^+[\frac{1}{b_1}]$ in $B[\frac{1}{b_1}]$. The Huber pair $(S^+[b_2, \ldots, b_n], S^+)$ is olivine by induction. But $\mathrm{Spa}(S, S^+)$ is the locus in $\mathrm{Spa}(S^+[b_2, \ldots, b_n], S^+)$ in which $\frac{1}{b_1} \neq 0$, so this one is also olivine. Since $\mathrm{Spa}(B, B^+) = \mathrm{Spa}(R, R^+) \cup \mathrm{Spa}(S, S^+)$ this finishes the proof that (B, B^+) is olivine. $\square$

Remark 1.2.21. *For an arbitrary Huber pair for which B^+ serves as a ring of definition we can consider the commutative diagram*

$$\begin{tikzcd}
\mid \mathrm{Spd}(B, B^+) \mid \arrow[r] \arrow[d] & \mathrm{Spo}(B, B^+) \arrow[d] \\
\varprojlim_i \mid \mathrm{Spd}(B_i, B^+) \mid \arrow[r] & \varprojlim_i \mathrm{Spo}(B_i)
\end{tikzcd}$$

where B_i ranges over all subrings of B that are of finite type. By proposition 1.2.20 the bottom horizontal arrow is a homeomorphism and one can verify directly that the right vertical arrow is also a homeomorphism. It is not clear to us if the left vertical arrow is a homeomorphism or not since taking limits of v-sheaf does not necessarily commute with taking underlying topological spaces. Adding to the complexity of the situation the transition maps $\mathrm{Spd}(B_i, B^+) \to \mathrm{Spd}(B_j, B^+)$ might not be quasi-compact. Any counterexample to question 1.2.19 should come from this failure.

Question 1.2.22. *Let $B = \mathbb{F}_p[T_1, \ldots, T_n, \ldots]$ be the free algebra in infinitely many variables endowed with the discrete topology and let $B^+ = \mathbb{F}_p$. Is (B, B^+) olivine?*

1.2.4 Some open and closed subsheaves of $\mathrm{Spd}(B, B^+)$

By [51] 12.9 open subsets of $\mathrm{Spo}(B, B^+)$ define open subsheaves of $\mathrm{Spd}(B, B^+)$, and when (B, B^+) is olivine this association is bijective. Since the formation of $\mathrm{Spd}(B, B^+)$ commutes with localization in $\mathrm{Spa}(B, B^+)$, one can compute the open subsheaf corresponding to classical localizations. The following lemma describes, in some cases, the open subsheaf associated to an analytic localizations .

Lemma 1.2.23. *Let (B, B^+) be a complete Huber pair over $\mathbb{Z}_p$ and suppose that B is adic with ideal of definition I. Let $b \in B$, let B_b be the completion of B with respect to the (b, I)-adic topology and let B_b^+ be the integral closure of $B^+ + (B_b)^{\circ\circ}$ in B_b. The open subsheaf associated to the analytic localization $N_{b<<1} \subseteq |\mathrm{Spd}(B, B^+)|$ is represented by $\mathrm{Spd}(B_b, {B_b}^+)$.*

Proof. If a map $f : \mathrm{Spa}(R, R^+) \to \mathrm{Spd}(B, B^+)$ factors through $\mathrm{Spd}(B_b, {B_b}^+)$ then $f^*(b)$ is topologically nilpotent in $R^\sharp$. This implies that $\mathrm{Spo}(f)(\mathrm{Spa}(R, R^+)) \subseteq N_{b<<1}$ and since this happens for every test space $(R, R^+) \in \mathrm{Perf}$, the map of v-sheaves $\mathrm{Spd}(B_b, {B_b}^+) \to \mathrm{Spd}(B, B^+)$ must factors through the subsheaf associated to $N_{b<<1}$. Moreover, since $B \subseteq B_b$ is dense, the maps $f^* : B_b \to R^\sharp$ are determined by their restriction to B, this implies that $\mathrm{Spd}(B_b, {B_b}^+) \to \mathrm{Spd}(B, B^+)$ is an injective map.

We must prove that if $f : \mathrm{Spa}(R^\sharp, {R^\sharp}^+) \to \mathrm{Spa}(B, B^+)$ is such that

$$\mathrm{Spo}(f)(\mathrm{Spa}(R^\sharp, {R^\sharp}^+)) \subseteq N_{b<<1},$$

then it factors through a (unique) map $\mathrm{Spa}(R^\sharp, {R^\sharp}^+) \to \mathrm{Spa}(B_b, {B_b}^+)$. Given a point $x \in \mathrm{Spa}(R^\sharp, {R^\sharp}^+)$ and a pseudo-uniformizer $\varpi \in {R^\sharp}^+$ we let x^a denote the rank 1 generization of x, we have that $|f^*b^n|_x \leq |\varpi|_x$ for some n since by hypothesis $|f^*b|_{x^a} < 1$.

Let $U = U(\frac{f^*b^n}{\varpi}) \subseteq \mathrm{Spa}(R^\sharp, {R^\sharp}^+)$. If $R_1 = \widehat{(R^{\sharp,+})[\frac{b^n}{\varpi}]}$ where we complete by the ϖ-adic topology, then $R' = H^0(U, \mathcal{O}_X) = R_1[\frac{1}{\varpi}]$ and $R'^+ = H^0(U, \mathcal{O}_X^+)$ is the integral closure of $R^{\sharp,+} + R_1^{\circ\circ}$ in R'. Since $f^*b^n \in R'^{\circ\circ}$ the map $B \to R'$ is continuous when B is given the (I, b)-adic topology. Moreover, since R'° is complete we get a map $B_b \to R'^\circ$. This gives a factorization $\mathrm{Spa}(R', R'^+) \to \mathrm{Spa}(B_b, {B_b}^+)$, and a map $\mathrm{Spa}(R', R'^+)^\flat \to \mathrm{Spd}(B_b, {B_b}^+)$. We have proved that locally f factors through $\mathrm{Spd}(B_b, {B_b}^+)$, since $\mathrm{Spd}(B_b, {B_b}^+)$ is a v-sheaf and the factorization is unique we may glue this to a map $\mathrm{Spa}(R, R^+) \to \mathrm{Spd}(B_b, {B_b}^+)$. ∎

In general the subsheaf $N_{b<<1}$ might not be of the form $\mathrm{Spd}(R, R^+)$ for a Huber pair (R, R^+), but it can always be described as the locus in which b is topologically nilpotent.

Recall that we defined $\mathrm{Spo}(B, B^+)^\dagger \subseteq \mathrm{Spo}(B, B^+)$ as the closed subset of bounded points. The definition of boundedness implies that $\mathrm{Spo}(B, B^+)^\dagger$ is stable under vertical generization and by proposition 1.1.17 it defines a closed subsheaf of $\mathrm{Spd}(B, B^+)$. Here is a different description.

Proposition 1.2.24. *Define* $\mathrm{Spd}(B, B^+)^\dagger : \mathrm{Perf} \to \mathrm{Sets}$ *to parametrize triples* $(R^\sharp, \iota, f)$ *where* $(R^\sharp, \iota)$ *is an untilt of* R *and* $f : \mathrm{Spa}(R^{\sharp,\circ}, R^{\sharp,+}) \to \mathrm{Spa}(B, B^+)$ *is a morphism of pre-adic spaces. We get the following Cartesian diagram*

$$\begin{tikzcd}
\mathrm{Spd}(B, B^+)^\dagger \arrow[r] \arrow[d] & \mathrm{Spd}(B, B^+) \arrow[d] \\
\underline{\mathrm{Spo}(B, B^+)^\dagger} \arrow[r] & \underline{\mathrm{Spo}(B, B^+)}
\end{tikzcd}$$

Proof. We first prove that $\mathrm{Spd}(B, B^+)^\dagger \to \mathrm{Spd}(B, B^+)$ is a closed immersion. Let $\mathbb{A}^{|B|}_{\mathbb{Z}_p}$ denote the functor sending $(R, R^+) \mapsto (R^\sharp, \iota, x)$ where $(R^\sharp, \iota)$ is an untilt and x is a tuple with values in $R^\sharp$ indexed by elements of B. The similarly defined space $\mathbb{A}^{|B|,\dagger}_{\mathbb{Z}_p}$ parametrizing tuples in $R^{\sharp,\circ}$ sits inside $\mathbb{A}^{|B|}_{\mathbb{Z}_p}$, and we have a basechange identity $\mathrm{Spd}(B, B^+)^\dagger = \mathbb{A}^{|B|,\dagger}_{\mathbb{Z}_p} \times_{\mathbb{A}^{|B|}_{\mathbb{Z}_p}} \mathrm{Spd}(B, B^+)$. Since the limit of closed immersions is a closed immersion we can reduce to prove that $\mathbb{A}^{1,\dagger}_{\mathbb{Z}_p} \to \mathbb{A}^1_{\mathbb{Z}_p}$ is a closed immersion. Consider the basechange by maps with source an affinoid perfectoid, $f_r : \mathrm{Spa}(R, R^+) \to \mathbb{A}^1_{\mathbb{Z}_p}$ with $r \in R^\sharp$. Then $\mathbb{A}^{1,\dagger} \times_{\mathbb{A}^1_{\mathbb{Z}_p}} \mathrm{Spa}(R, R^+)$ is the complement in $\mathrm{Spa}(R^\sharp, R^{\sharp+})$ of

$$\bigcup_{\varpi} \{x \in \mathrm{Spa}(R^\sharp, R^{\sharp+}) \mid |1|_x \leq |r \cdot \varpi|_x \neq 0\}$$

where ϖ ranges over elements of $R^{\sharp,\circ\circ}$. This is a closed subset stable under vertical generization and defines a closed immersion into $\mathrm{Spa}(R, R^+)$ as we wanted to show.

Once we know $\mathrm{Spd}(B, B^+)^\dagger$ and $\mathrm{Spd}(B, B^+) \times_{\underline{\mathrm{Spo}(B,B^+)}} \underline{\mathrm{Spo}(B, B^+)^\dagger}$ are closed immersions it suffices to verify on geometric points that they agree. Let $q : \mathrm{Spa}(C, C^+) \to \mathrm{Spd}(B, B^+)$ be a geometric point with rank 1 generization q^a. That q maps to a bounded point in $\mathrm{Spo}(B, B^+)$ means by definition that $|B|_{q^a} \leq 1$, which is precisely the condition that the map $B \to C^\sharp$ factors through $O_{C^\sharp}$. $\square$

Remark 1.2.25. *One should be careful with the notion of bounded points since this notion is not compatible with localization (and it shouldn't). If* $\mathrm{Spa}(R, R^+) \subseteq \mathrm{Spa}(B, B^+)$ *is a rational localization and* $x \in \mathrm{Spo}(R, R^+)$ *it might happen that* $x \in \mathrm{Spo}(B, B^+)^\dagger \cap \mathrm{Spo}(R, R^+)$ *but* $x \notin \mathrm{Spo}(R, R^+)^\dagger$*. For example,* $(\mathbb{Z}_p^\diamondsuit)^\dagger = \mathbb{Z}_p^\diamondsuit$ *but* $(\mathbb{Q}_p^\diamondsuit)^\dagger = \emptyset$*.*

Lemma 1.2.26. *Suppose that (A, A^+) and (B, B^+) are complete Huber pairs over $\mathbb{Z}_p$ and that we have an adic homomorphism $(B, B^+) \to (A, A^+)$. Then the induced map of v-sheaves $\mathrm{Spd}(A, A^+)^\dagger \to \mathrm{Spd}(B, B^+)^\dagger$ is representable in spatial diamonds and in particular qcqs.*

Proof. Since the map $(B, B^+) \to (A, A^+)$ is adic we can write (A, A^+) as a filtered colimit $\varinjlim_{i\in I}(A_i, {A_i}^+)$ where each $(A_i, {A_i}^+)$ is topologically of finite type over (B, B^+), and the transition maps realize $A_i \to A_j$ as a topological subrings for $i < j$. One can see directly that $\mathrm{Spd}(A, A^+)^\dagger = \varprojlim_i \mathrm{Spd}(A_i, {A_i}^+)^\dagger$ and by [51] 12.17 it is enough to prove that $\mathrm{Spd}(A_i, {A_i}^+)^\dagger \to \mathrm{Spd}(B, B^+)^\dagger$ is representable in spatial diamonds. By definition of being topologically of finite type there is a family of sets $M = \{M_i\}_{i=1}^n$ with $B \cdot M_i$ an open ideal and a strict surjection $B\langle T_1, \dots T_n\rangle_{M_1,\dots,M_n} \to A_i$ compatible with rings of integral elements. One can verify directly that the induced map

$$\mathrm{Spd}(A_i, {A_i}^+) \to \mathrm{Spd}(B\langle T_1, \dots T_n\rangle_{M_1,\dots,M_n}, {B\langle T_1, \dots T_n\rangle_{M_1,\dots,M_n}}^+)$$

is a closed immersion and that $\mathrm{Spd}(A_i, {A_i}^+)^\dagger$ is the basechange of the corresponding bounded subsheaf. Since closed immersions are representable in spatial diamonds it is enough to verify the case in which $A_i = B\langle T_1\rangle_{M_1}$. There is an open immersion $\mathrm{Spd}(B\langle T_1\rangle_{M_1}, {B\langle T_1\rangle_{M_1}}^+) \to \mathbb{A}^1_B$ and such that $\mathrm{Spd}(B\langle T_1\rangle_{M_1}, {B\langle T_1\rangle_{M_1}}^+) \cap \mathbb{A}^{1,\dagger}_B = \mathrm{Spd}(B\langle T_1\rangle_{M_1}, {B\langle T_1\rangle_{M_1}}^+)^\dagger$. To prove $\mathbb{A}^{1,\dagger} \to \mathrm{Spd}(B, B^+)^\dagger$ is representable in spatial diamonds it is enough to verify that basechanges by affinoid perfectoid are spatial diamonds. The basechange of a map $\mathrm{Spa}(R, R^+) \to \mathrm{Spd}(B, B^+)^\dagger$ is representable by $\mathrm{Spd}(R^\sharp\langle T\rangle, R')$ where R' is the minimal ring of integral elements containing $R^{\sharp,+}$. Since $R^\sharp\langle T\rangle$ is Tate this space is a spatial diamond. $\square$

The following statement says that at least the bounded locus of a Huber pair is always olivine.

Proposition 1.2.27. *Suppose that (B, B^+) is a complete Huber pair over $\mathbb{Z}_p$. The natural map*

$$|\mathrm{Spd}(B, B^+)^\dagger| \to \mathrm{Spo}(B, B^+)^\dagger$$

is a homeomorphism.

Proof. Let $B_0 \subseteq B^+$ be a ring of definition and express (B, B^+) as a filtered colimit of Huber pairs of the form $\varinjlim_{i\in J}(B_i, B_i^+)$ where both B_i and B_i^+ are of finite type over B_0. We have that $\mathrm{Spd}(B, B^+)^\dagger = \varprojlim \mathrm{Spd}(B_i, {B_i}^+)^\dagger$. By proposition 1.2.20 each $(B_i, {B_i}^+)$ is olivine and by lemma 1.2.26 the transition maps are representable in spatial diamonds. Let $\mathbb{D}^\times_{B_i^\dagger}$ denote the punctured open unit disc over $\mathrm{Spd}(B_i, {B_i}^+)^\dagger$, and let π_i be the projection map $\pi_i : \mathbb{D}^\times_{B_i^\dagger} \to \mathrm{Spd}(B_i, {B_i}^+)^\dagger$. Observe that π_i is universally open since it is ℓ-cohomologically smooth. We have that $\mathbb{D}^\times_{B_0}$ is a locally spatial diamond represented by $(\mathrm{Spa}(B_0[[t]], B_0[[t]])^{t\neq 0})^\diamond$. In particular, $\mathbb{D}^\times_{B_i^\dagger}$ is also a locally spatial diamond and since the transition maps $\mathbb{D}^\times_{B_i^\dagger} \to \mathbb{D}^\times_{B_j^\dagger}$ are qcqs we see that by [51] 12.17 $|\mathbb{D}^\times_{B^\dagger}| = \varprojlim |\mathbb{D}^\times_{B_i^\dagger}|$. It is enough to prove that $\pi : |\mathbb{D}^\times_{B^\dagger}| \to$

$\mathrm{Spo}(B, B^+)^\dagger$ is a quotient map. Let $S \subseteq \mathrm{Spo}(B, B^+)^\dagger$ with $\pi^{-1}(S)$ open. For every point $y \in \pi^{-1}(S)$ there is an index $j_y \in J$ and an open subset of $U_y \subseteq \mathbb{D}^\times_{B_i^\dagger}$ whose preimage in $\mathbb{D}^\times_{B^\dagger}$ is contained in $\pi^{-1}(S)$ and contains y. Now, $\pi_{j_y}(U_y)$ is open in $|\mathrm{Spd}(B_i, {B_i}^+)^\dagger|$ and since $(B_i, {B_i}^+)$ is olivine it is also open in $\mathrm{Spo}(B_i, {B_i}^+)^\dagger$. The preimage of $\pi_{j_y}(U_y)$ in $\mathrm{Spo}(B, B^+)^\dagger$ contains $\pi(y)$, is open and it is contained in $h^{-1}(S)$ which finishes the proof that S is open in $\mathrm{Spo}(B, B^+)^\dagger$. □

1.2.5 Discrete Huber pairs in characteristic p

In the following section, when we discuss the reduction functor, we will need to understand the olivine spectrum of Huber pairs associated to perfect schemes. For this reason we discuss this case in detail. For the rest of the subsection A denotes a discrete perfect ring in characteristic p and $A^+ \subseteq A$ is integrally closed.

Proposition 1.2.28. *Let (A, A^+) be as above. The projection map*

$$\mathrm{Spo}(A, A^+)^\dagger \to \mathrm{Spa}(A, A^+)$$

is surjective. Moreover, if $\mathfrak{S} \subseteq \mathrm{Spa}(A, A^+)$ is stable under arbitrary generization and $h^{-1}(\mathfrak{S})$ is open in $\mathrm{Spo}(A, A^+)^\dagger$ then $\mathfrak{S}$ is open in $\mathrm{Spa}(A, A^+)$.

Proof. The complement of the bounded locus consists of d-analytic points. Since A has the discrete topology every d-analytic point is meromorphic. If $x \in \mathrm{Spo}(A, A^+)$ is a meromorphic point it has a meromorphic specialization $y \in \mathrm{Spo}(A, A^+)$ with $h(x) = h(y)$, y is non-analytic and in particular bounded. This gives $h(\mathrm{Spo}(A, A^+)) = h(\mathrm{Spo}(A, A^+)^\dagger)$ which is $\mathrm{Spa}(A, A^+)$.

For the second claim observe that the map $\mathrm{Spd}(A((t)), A^+ + t \cdot A[[t]]) \to \mathrm{Spd}(A, A^+)$ surjects onto $\mathrm{Spd}(A, A^+)^\dagger$ and represents the punctured open unit ball over it. Let f denote the map of adic spaces $f : \mathrm{Spa}(A((t)), A^+ + t \cdot A[[t]]) \to \mathrm{Spa}(A, A^+)$, it is enough to prove that if $\mathfrak{S}$ is stable under generization and $f^{-1}(\mathfrak{S})$ is open then $\mathfrak{S}$ is open. The rest of the argument is an easier version of the proof of proposition 1.2.17. In this case one exploits the constructible topology of $\mathrm{Spa}(A((t)), A^+ + t \cdot A[[t]])$.

Let $x \in \mathfrak{S}$, and let $y \in \mathrm{Spa}(A((t)), A^+ + t \cdot A[[t]])$ a lift of x. For every open $U_{x,a_1,a_2} = \{x \in \mathrm{Spa}(A, A^+) \mid |a_1|_x \leq |a_2|_x \neq 0\}$ we choose $n \in \mathbb{N}$ such that $|t^n|_y \leq |a_2|_y$ and define U_{y,a_1,a_2} as

$$\{z \in \mathrm{Spa}(A((t)), A^+ + t \cdot A[[t]]) \mid |a_1|_z, |t^n|_z \leq |a_2|_z \neq 0\}.$$

Observe that $y \in U_{y,a_1,a_2}$, that it is quasi-compact and that $f(U_{y,a_1,a_2}) = U_{x,a_1,a_2}$. Notice that since $\mathfrak{S}$ is stable under generization in $\mathrm{Spa}(A, A^+)$ the intersection of all U_{x,a_1,a_2} that contain x is contained in $\mathfrak{S}$. This implies that the intersection of the U_{y,a_1,a_2} is contained in $f^{-1}(\mathfrak{S})$. Since $f^{-1}(\mathfrak{S})$ is open, by the usual compactness argument in the patch topology there is a finite subset with $\bigcap U_{y,a_1,a_2} \subseteq f^{-1}(\mathfrak{S})$. We denote by Z this intersection and we observe that $f(Z) = \bigcap U_{x,a_1,a_2}$ which is open in $\mathrm{Spa}(A, A^+)$ and contains x. □

Proposition 1.2.29. *If A and A^+ are discrete perfect valuation rings with the same fraction field then (A, A^+) is olivine.*

Proof. If $\mathrm{Spo}(A, A^+)^\dagger = \mathrm{Spo}(A, A^+)$ then proposition 1.2.27 proves that (A, A^+) is olivine. Suppose $x \in \mathrm{Spo}(A, A^+) \setminus \mathrm{Spo}(A, A^+)^\dagger$, then x is meromorphic and there is $a \in A$ with $1 < |a|_x^a$. We must have $\frac{1}{a} \in A^+$ since A^+ is a valuation ring and $a \notin A^+$. Let $b = \frac{1}{a}$, we claim that $A = A^+[\frac{1}{b}]$. By the Archimedean property of $|\cdot|_x^a$ for every $a' \in A$ there is a big enough $n \in \mathbb{N}$ with $|b^n \cdot a'|_x^a < 1$. Since A^+ is a valuation ring either $a' \cdot b^n \in A^+$ or $\frac{1}{a' \cdot b^n} \in A^+$, but the second case contradicts that $|\cdot|_x^a \in \mathrm{Spa}(A, A^+)$.

By proposition 1.2.17 (A^+, A^+) is olivine and since $\mathrm{Spo}(A, A^+) \subseteq \mathrm{Spo}(A^+, A^+)$ is the open locus in which $b \neq 0$ we conclude by proposition 1.2.8 that (A, A^+) is also olivine. □

In what follows we prove some lemmas to prepare the proof of theorem 4. As we have mentioned in the introduction the statements and techniques are derivative of Scholze and Weinstein's full faithfulness result [53] 18.3.1.

Lemma 1.2.30. *Let (A, A^+) be a non-analytic perfect Huber pair in characteristic p. Then there is a unique morphism of v-sheaves $\mathrm{Spd}(A, A^+) \to \mathbb{Z}_p^\diamondsuit$. It is given by the composition $\mathrm{Spd}(A, A^+) \to \mathbb{F}_p^\diamondsuit \to \mathbb{Z}_p^\diamondsuit$.*

Proof. It is enough to prove that for every geometric point $\mathrm{Spa}(C, C^+) \to \mathrm{Spd}(A, A^+)$ the composition to $\mathbb{Z}_p^\diamondsuit$ factors through $\mathbb{F}_p^\diamondsuit$. Consider the product of points $\mathrm{Spa}(R_\infty, {R_\infty}^+) \to \mathrm{Spd}(A, A^+)$ as in the proof of lemma 1.2.16, with $R_\infty^+ = \prod_{i=1}^\infty C^+$ and $\varpi_\infty = (\varpi^{p^i})$. The composition $\mathrm{Spa}(R_\infty, {R_\infty}^+) \to \mathrm{Spd}(A, A^+) \to \mathbb{Z}_p^\diamondsuit$ defines an untilt of R_∞ given by an element $\xi = p + (\varpi_\infty)^{\frac{1}{p^k}} \cdot \alpha$ with $\alpha \in W(R^+)$. For any $i \in \mathbb{N}$ the projection map $\iota_i : R_\infty \to C$ defines an untilt of C and since the composition $(A, A^+) \to (R_\infty, {R_\infty}^+) \xrightarrow{\iota_i} (C, C^+)$ is independent of the projection map chosen, all of these untilts agree. This says that the ideal I_i generated by $\iota_i(\xi)$ in $W(C^+)$ agree, we call this ideal I. Since $\iota_i(\xi) = p - \varpi^{\frac{p^i}{p^k}} \iota_i(\alpha)$ the sequence $\iota_i(\xi)$ converges to p in the (p, ϖ)-adic topology. But the ideal associated to an untilt is closed, so $p \in I$ and $\mathrm{Spa}(C, C^+) \to \mathbb{Z}_p^\diamondsuit$ factors through $\mathbb{F}_p^\diamondsuit$. □

Lemma 1.2.31. *Let (A, A^+) be as above and let (B, B^+) be a complete Huber pairs over $\mathbb{Z}_p$. Then every morphism of v-sheaves $\mathrm{Spd}(A, A^+) \to \mathrm{Spd}(B, B^+)$ comes from a unique morphism of Huber pairs $(B, B^+) \to (A, A^+)$.*

Proof. Given a map $\mathrm{Spd}(A, A^+) \to \mathrm{Spd}(B, B^+)$ we associate to it a map of pre-adic spaces $\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(B, B^+)$. Let $S = \mathrm{Spd}(A, A^+)$, $R = A((t^{\frac{1}{p^\infty}}))$, $R^+ = A^+ + (t^{\frac{1}{p^\infty}})A[[t^{\frac{1}{p^\infty}}]]$, $X = \mathrm{Spa}(R, R^+)$ and $X' = X \times_S X$. Notice that X is an affinoid perfectoid space and X' is a perfectoid space (which is not affinoid). The natural map $X \to \mathrm{Spd}(A, A^+)$ surjects onto $\mathrm{Spd}(A, A^+)^\dagger$ and we get an equalizer diagram:

$$0 \to Hom(\mathrm{Spd}(A, A^+)^\dagger, \mathrm{Spd}(B, B^+)) \to Hom(X, \mathrm{Spd}(B, B^+)) \rightrightarrows Hom(X', \mathrm{Spd}(B, B^+)).$$

Since X is affinoid perfectoid a homomorphism $f : X \to \mathrm{Spd}(B, B^+)$ is given by an untilt $R^\sharp$ and a continuous ring map $f^* : B \to R^\sharp$ such that $f^*(B^+) \subseteq R^{+,\sharp}$. By lemma 1.2.30 the untilt must be R.

A necessary condition for such a morphism to glue, is that $f^*(B)$ must be invariant under any automorphism of R over A. In particular, we may replace t by any topological nilpotent unit in R without changing the image of $f^*(B)$. Take an element $b \in B$, we want to show $f^*(b) \in A$. Observe that $t^{p^n} \cdot f^*(b)$ is topologically nilpotent for big enough n. Replacing by $t \mapsto t^{\frac{1}{p^m}}$ we conclude that $t^{p^n} f^*(b)$ is topologically nilpotent for all $n \in \mathbb{Z}$. This proves $f^*(b)$ is power-bounded which gives $f^*(b) \in A[[t^{\frac{1}{p^\infty}}]]$. We can write $f^*(b)$ as $a_0 + t^{\frac{1}{p^m}} q$ with $a_0 \in A$ and $q \in A[[t^{\frac{1}{p^\infty}}]]$. Since the second term converges to 0 under the substitution $t \mapsto t^{p^n}$ we see that $f^*(b) = a_0$. This defines a map of rings $B \to A$. Since the subspace topology of A in R is the discrete topology this ring morphism is continuous if and only if the original one was. Finally, we see that B^+ maps to $R^+ \cap A$ which is easily seen to be A^+. So far we have constructed a map $\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(B, B^+)$ with the property that the induced map $\mathrm{Spd}(A, A^+) \to \mathrm{Spd}(B, B^+)$ agrees with our original map in the locus $\mathrm{Spd}(B, B^+)^\dagger$. We wish to prove that the two maps agree.

Consider the map $\mathrm{Spd}(A, A^+) \to \mathrm{Spd}(B, B^+) \times \mathrm{Spd}(B, B^+)$ we will show that $\mathrm{Spd}(A, A^+)$ factors through the diagonal embedding $\Delta : \mathrm{Spd}(B, B^+) \to \mathrm{Spd}(B, B^+) \times \mathrm{Spd}(B, B^+)$. This can be verified on geometric points $\mathrm{Spd}(C, C^+) \to \mathrm{Spd}(A, A^+)$, and since the maps already agree on $\mathrm{Spd}(A, A^+)^\dagger$ it is enough to verify this on meromorphic points $x \in \mathrm{Spo}(A, A^+)$. Pick a pseudo-uniformizer $\varpi \in C$ and consider the product of points R_∞ as in the proof lemma 1.2.16 together with the map $\mathrm{Spa}(R_\infty, {R_\infty}^+) \to \mathrm{Spd}(A, A^+)$ given by the diagonal morphism $A \to \prod C$. Recall that for the product of points constructed in this way we have that

$$C \subseteq_\Delta R_\infty \subseteq \prod_{i=1}^{\infty} C,$$

and that although the diagonal embedding $C \subseteq_\Delta R_\infty$ is not continuous the composition $A \to R_\infty$ is continuous since A has the discrete topology.

The composition $\mathrm{Spa}(R_\infty, {R_\infty}^+) \to \mathrm{Spd}(A, A^+) \to \mathrm{Spd}(B, B^+) \times \mathrm{Spd}(B, B^+)$ gives two morphisms $f_1, f_2 : B \to R_\infty$, and one verifies that at the level of rings they both have to factor through the diagonal $C \subseteq_\Delta R_\infty$. The connected components of $\mathrm{Spa}(R_\infty, {R_\infty}^+)$ are in bijection with ultrafilters of $\mathbb{N}$. By the proof of lemma 1.2.16 the residue field at a non-principal ultrafilter $\mathcal{U}$ maps to the meromorphic specialization x_{mer}. Since $\mathrm{Spa}(C_\mathcal{U}, {C_\mathcal{U}}^+) \to \mathrm{Spd}(B, B^+) \times \mathrm{Spd}(B, B^+)$ factors through $\mathrm{Spd}(A, A^+)^\dagger$ (being non-analytic in $\mathrm{Spo}(A, A^+)$), it also factors through the diagonal. These maps are given by the composition $f_i : B \to C \to R_\infty \to C_\mathcal{U}$. We can conclude $f_1 = f_2$ since the map $C \to C_\mathcal{U}$ is injective. $\square$

We are no ready to prove theorem 4, which is the global version of lemma 1.2.31. For the convenience of the reader we state it again.

Theorem 1.2.32. *Let Y be a perfect non-analytic adic space over $\mathbb{F}_p$ and let X be a pre-adic* *space over $\mathbb{Z}_p$. The natural map*

$$Hom_{_{PreAd}}(Y, X) \to Hom(Y^\diamondsuit, X^\diamondsuit)$$

is bijective. In particular, $\diamondsuit$ is fully faithful when restricted to the category of perfect non-analytic adic spaces over $\mathbb{F}_p$.

This theorem says, intuitively speaking, that (up to perfection) one does not get new morphisms of v-sheaves when the source is a non-analytic adic space.

Proof. It is not hard to prove injectivity. For surjectivity, the hard part is to prove that a morphism $g : X^\diamondsuit \to Y^\diamondsuit$ induces a unique map of topological spaces $f : |X| \to |Y|$ that makes the following diagram commutative.

$$\begin{array}{ccc} \mid Y^\diamondsuit \mid & \xrightarrow{g} & \mid X^\diamondsuit \mid \\ \downarrow & & \downarrow \\ \mid Y \mid & \xrightarrow{f} & \mid X \mid \end{array}$$

Assume for the moment that this is the case. Let $U = \coprod_{i\in I} \mathrm{Spa}(B_i, {B_i}^+)$ be an open cover for X and let $V = \coprod_{j\in J} \mathrm{Spa}(B'_j, {B'_j}^+)$ be an open cover of $U \times_X U$. Given a map $Y^\diamondsuit \to X^\diamondsuit$ we can pullback U and V through f to obtain an open cover of adic spaces

$$Y_V \rightrightarrows Y_U \to Y$$

satisfying $Y_U^\diamondsuit = Y^\diamondsuit \times_{X^\diamondsuit} U^\diamondsuit$ and $Y_V^\diamondsuit = Y^\diamondsuit \times_{X^\diamondsuit} U^\diamondsuit$. It is enough to prove that $Y_U^\diamondsuit \to U^\diamondsuit$ and $Y_V^\diamondsuit \to V^\diamondsuit$ come from morphisms of pre-adic spaces. This follows from lemma 1.2.31 by a standard glueing argument, since U and V are a disjoint union of affinoid pre-adic spaces.

Verifying that $g : |Y^\diamondsuit| \to |X^\diamondsuit|$ descends to a continuous map $f : |Y| \to |X|$ can be done locally on $|Y|$, we may assume $Y = \mathrm{Spa}(A, A^+)$. For $y \in |Y|$ and $z \in \mathrm{Spo}(A, A^+)$ with $h(z) = y$, we define $f(y) := h(g(z))$. We must verify that this doesn't depend of the choice of z and that it is continuous. The map f is well defined if and only if $h(g(z)) = h(g(z_{mer}))$ when z is meromorphic, and by proposition 1.2.28 to prove continuity it is enough to prove that if $\mathfrak{S} \subseteq |X|$ is open then $f^{-1}(\mathfrak{S})$ is stable under arbitrary generization in $\mathrm{Spa}(A, A^+)$. Let $w \in \mathrm{Spa}(A, A^+)$ a horizontal generization of y. Let $(k_y, {k_y}^+)$ and $(k_w, {k_w}^+)$ denote the affinoid residue fields of w and y and let K_w denote the smallest ring containing k_w^+ and A. It is enough to prove that the induced maps $|\mathrm{Spd}(K_w, k_w^+)| \to |X^\diamondsuit|$ and $|\mathrm{Spd}(k_y, k_y^+)| \to |X^\diamondsuit|$ descend to continuous maps $|\mathrm{Spa}(K_w, k_w^+)| \to |X|$ and $|\mathrm{Spa}(k_y, k_y^+)| \to |X|$. In summary, we have reduced the initial claim to the case in which $Y = \mathrm{Spa}(A, A^+)$ where $A^+ \subseteq A$ and the two rings are perfect non-analytic valuation rings that have the same fraction field.

Lemma 1.2.33. *Let X be a pre-adic space over $\mathbb{Z}_p$ as above and let $Y = \mathrm{Spa}(A, A^+)$ where A^+ and A are perfect non-analytic valuation rings that have the same fraction field, and let $g : \mathrm{Spd}(A, A^+) \to X^\diamondsuit$ be a map. Let $h(c) \in \mathrm{Spa}(A, A^+)$ denote the unique closed point and let $c \in \mathrm{Spo}(A, A^+)$ denote the unique non-analytic point mapping to $h(c)$. If $h(g(c)) \in |X|$*

lies in $\mathrm{Spa}(B_1, {B_1}^+) \subseteq X$ *then g factors through a map* $\mathrm{Spd}(A, A^+) \to \mathrm{Spd}(B_1, {B_1}^+) \subseteq X^\lozenge$. *In particular, g is coming from a map of pre-adic spaces* $\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(B_1, {B_1}^+) \subseteq X$.

Proof. Suppose to get a contradiction that there is an "exotic" g that does not satisfy this property. By proposition 1.2.29 (A, A^+) is olivine so we may treat $|\mathrm{Spd}(A, A^+)|$ and $\mathrm{Spo}(A, A^+)$ as the same object. Let $U_1 \subseteq \mathrm{Spo}(A, A^+)$ be the open subset associated to the pullback of $\mathrm{Spd}(B_1, {B_1}^+)$, this is by assumption a proper open subset. Let $Z = \mathrm{Spo}(A, A^+) \setminus U_1$, it is a quasi-compact topological space and we may use [53] 18.3.2 to find the largest prime $\mathfrak{p}_m \in \mathrm{Spec}(A)$ that is the support of an element in Z. Replacing A and A^+ by $A/\mathfrak{p}_m$ and $A^+/\mathfrak{p}_m$ we may assume that all elements of $z \in Z$ satisfy $supp(z) = 0$, we let $K = Frac(A)$. In this case $Z \subseteq \mathrm{Spo}(K, A^+)$. Since Z is a closed subset it contains the unique closed point q of $\mathrm{Spo}(K, A^+)$, this is the unique non-analytic point such that $h(q)$ is closed in $\mathrm{Spa}(K, A^+)$. We claim that U_1 contains every analytic localization of the form $N_{n<<1}$ with $n \neq 0$ and $n \in supp(c)$. Indeed, if $z \in Z \cap N_{n<<1}$ then $|n|^a_z < 1$ and either z or its formal specialization would have non-trivial support contradicting the assumption that $\mathfrak{p}_m$ was the largest.

The composition $\mathrm{Spd}(K, A^+) \to X^\lozenge$ must also factor through some other open affine subsheaf $\mathrm{Spd}(B_2, {B_2}^+)$, since it has a unique closed point. We let U_2 be the open in $\mathrm{Spo}(A, A^+)$ associated to the pullback of $\mathrm{Spd}(B_2, {B_2}^+)$. By example 1.2.9 there is an open neighborhood of q of the form $U_{0\leq b\neq 0} \cap U_{1\leq b'\neq 0} \cap N_{n<<1}$ and contained in U_2. Moreover, in example 1.2.9 q corresponds to the triple of prime ideals $(0, 0, \mathfrak{m})$ where $\mathfrak{m}$ denotes the maximal ideal of A^+. With this description it is easy to see that $U_{1\leq b'\neq 0} = \mathrm{Spo}(A, A^+)$ and that $N_{n<<1} = \mathrm{Spo}(A, A^+)$ since they contain q. In summary, there is a classical localization $U_{0\leq b\neq 0}$ containing q and contained in U_2.

We have found neighborhoods $N_{b<<1} \subseteq U_1$ and $U_{0\leq b\neq 0} \subseteq U_2$ with $N_{b<<1} \cap U_{0\leq b\neq 0} \subseteq U_1 \cap U_2$, observe that $\mathrm{Spo}(A, A^+) = N_{b<<1} \cup U_{0\leq b\neq 0}$. Let A_b and A_b^+ denote the (b)-adic completion of A and A^+ respectively. Lemma 1.2.23 shows that $N_{b<<1}$ is represented by $\mathrm{Spd}(A_b, {A_b}^+)$. We also have that $U_{0\leq b\neq 0}$ is represented by $\mathrm{Spd}(A[\frac{1}{b}], A^+)$ and that the intersection $N_{b<<1} \cap U_{0\leq b\neq 0}$ is represented by $\mathrm{Spa}(A_b[\frac{1}{b}], A_b^+)$, notice that this last one is a perfectoid field. We let q_b denote the closed point of $N_{b<<1} \cap U_{0\leq b\neq 0}$ which is meromorphic in $\mathrm{Spo}(A, A^+)$.

Since these morphisms glue, there is an affinoid open subspace

$$\mathrm{Spa}(B_3, {B_3}^+) \subseteq \mathrm{Spa}(B_1, {B_1}^+) \times_X \mathrm{Spa}(B_2, {B_2}^+)$$

and a map $\mathrm{Spa}(A_b[\frac{1}{b}], A_b^+) \to \mathrm{Spd}(B_3, {B_3}^+)$ making the following diagram commutative:

$$\begin{tikzcd}
\mathrm{Spa}(A_b[\frac{1}{b}], A_b^+) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathrm{Spd}(A[\frac{1}{b}], A^+) \arrow[d] \\
& \mathrm{Spd}(B_3, {B_3}^+) \arrow[r] \arrow[d] & \mathrm{Spd}(B_2, {B_2}^+) \arrow[d] \\
\mathrm{Spd}(A_b, A_b^+) \arrow[r] & \mathrm{Spd}(B_1, {B_1}^+) \arrow[r] & X^\lozenge
\end{tikzcd}$$

By lemma 1.2.31 the map $\mathrm{Spd}(A[\frac{1}{b}], A^+) \to \mathrm{Spd}(B_2, {B_2}^+)$ is given by a map of Huber pairs $(B_2, {B_2}^+) \to (A[\frac{1}{b}], A^+)$. Since

$$\mathrm{Spo}(B_3, {B_3}^+) \subseteq \mathrm{Spo}(B_2, {B_2}^+)$$

is of the form $h^{-1}(\mathrm{Spa}(B_3, {B_3}^+))$ the pullback of $\mathrm{Spo}(B_3, {B_3}^+)$ to $\mathrm{Spo}(A[\frac{1}{b}], A^+)$ has the form $h^{-1}(U_3)$ for some $U_3 \subseteq \mathrm{Spa}(A[\frac{1}{b}], A^+)$. Moreover, $h(q_b)$ is the closed point of $\mathrm{Spa}(A[\frac{1}{b}], A^+)$. This proves that $\mathrm{Spd}(A[\frac{1}{b}], A^+)$ factors through $\mathrm{Spd}(B_3, {B_3}^+)$ and consequently through $\mathrm{Spd}(B_1, {B_1}^+)$ contradicting our initial assumption.

□

□

We now study perfect non-analytic Huber pairs of the form (A, A). These are the type of Huber pairs that we will associate to perfect affine schemes to develop our theory of specialization.

Proposition 1.2.34. *Let A be a ring endowed with the discrete topology and $f^* : (B, B^+) \to (A, A)$ a map of Huber pairs, then the following hold:*

1. $\mathrm{Spo}(f)$ *is saturated with respect to h, in other words*

$$\mathrm{Spo}(f)(\mathrm{Spo}(A, A)) = h^{-1}(f(\mathrm{Spa}(A, A))).$$

2. $f(\mathrm{Spa}(A, A))$ *is stable under horizontal specializations in* $\mathrm{Spa}(B, B^+)$.

3. $f(\mathrm{Spa}(A, A))$ *is stable under vertical generization in* $\mathrm{Spa}(B, B^+)$.

Proof. To prove the first claim let $y \in \mathrm{Spo}(A, A)$ and let $x = \mathrm{Spo}(f)(y)$. If x is d-analytic, y must be meromorphic and y_{mer} maps to x_{mer} under $\mathrm{Spo}(f)$, that is $h^{-1}(h(x)) \subseteq Im(\mathrm{Spo}(f))$.

Suppose now that x is non-analytic and that x^{mer} exists in $\mathrm{Spo}(B, B^+)$. In this case, y^{mer} might not exist if $|\cdot|^h_y$ is not microbial, and even when y^{mer} exists it may not be true that y^{mer} maps to x^{mer} under $\mathrm{Spo}(f)$. Indeed, it may happen that y^{mer} maps to x instead. For these reasons we use a different construction.

Consider $h(y) \in \mathrm{Spa}(A, A)$ together with its residue field map $\iota_{h(y)} : \mathrm{Spa}(K_{h(y)}, {K_{h(y)}}^+) \to \mathrm{Spa}(A, A)$. Notice that $\iota_{h(y)}$ factors through a map $g : \mathrm{Spa}(K^+_{h(y)}, K^+_{h(y)}) \to \mathrm{Spa}(A, A)$, so it is enough to prove that x^{mer} is in the image of $\mathrm{Spo}(f \circ g)$. Take an element $b \in B$ with $|b|^a_{x^{mer}} \notin \{0, 1\}$ and replace it by its inverse in $K_{h(y)}$, whenever it is necessary, so that $b \in K^+_{h(y)}$. Define K^+ as the (b)-adic completion of $K^+_{h(y)}$, and let $K = K^+[\frac{1}{b}]$. We get the following commutative diagram,

$$\begin{tikzcd}
\mathrm{Spa}(K, K^+) \arrow[rr] \arrow[dr] & & \mathrm{Spo}(K^+_{h(y)}, K^+_{h(y)}) \arrow[dl] \\
& \mathrm{Spo}(B, B^+) &
\end{tikzcd}$$

and one can verify that the image of the closed point of $\mathrm{Spa}(K, K^+)$ in $\mathrm{Spo}(B, B^+)$ is x^{mer}. This proves that $h^{-1}(h(x)) \subseteq Im(\mathrm{Spo}(f))$.

The proof of the second claim also follows from observing that the residue field map $\iota_{h(y)}$ factors through g. Indeed, we get the following commutative diagram of adic spaces:

$$\begin{tikzcd}
\mathrm{Spa}(K_{h(y)}, {K_{h(y)}}^+) \arrow[r] \arrow[d] & \mathrm{Spa}(K^+_{h(y)}, K^+_{h(y)}) \arrow[r, "g"] \arrow[d] & \mathrm{Spa}(A, A) \arrow[d] \\
\mathrm{Spa}(K_{h(x)}, {K_{h(x)}}^+) \arrow[r] & \mathrm{Spa}(K^+_{h(x)}, K^+_{h(x)}) \arrow[r, "g'"] & \mathrm{Spa}(B, B^+)
\end{tikzcd}$$

Where we use that $K^+_{h(x)} = K^+_{h(y)} \cap K_{h(x)}$ to define g'. Moreover, the vertical map on the left is surjective since $h(x) = f(h(y))$ and one can deduce that the vertical map in the middle column is also surjective because the map of valuation rings is local. A prime ideal of $J \subseteq K^+_{h(x)}$ determines a horizontal specializations of $|\cdot|_{h(x)}$, namely $|\cdot|_{h(x)}/J$, and every horizontal specialization of $h(x)$ can be constructed in this way. For J as above we let $K_J^+ = K^+_{h(x)}/J$ and $K_J = Frac(K_J^+)$, we get the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Spa}(K_J, {K_J}^+) \arrow[r] \arrow[drr, bend right] & \mathrm{Spa}(K_J^+, K_J^+) \arrow[rrr] \arrow[dr] & & & \mathrm{Spa}(K^+_{h(y)}, K^+_{h(y)}) \arrow[dll] \\
& & \mathrm{Spa}(B, B^+) & &
\end{tikzcd}$$

The closed point of $\mathrm{Spa}(K_J, {K_J}^+)$ maps to the horizontal specialization of $h(x)$ associated to the ideal J.

The third claim follows from corollary 1.2.15 and from the first claim. $\square$

Definition 1.2.35. *We say that a subset of* $\mathrm{Spo}(B, B^+)$ *is a schematic subset if it is a union of sets of the form* $\mathrm{Spo}(m)(\mathrm{Spo}(A, A))$ *where* (A, A) *is given the discrete topology and* $m^* : (B, B^+) \to (A, A)$ *is a map of Huber pairs.*

Proposition 1.2.36. *Suppose that* $Z \subseteq \mathrm{Spo}(B, B^+)$ *is a schematic closed subset. Let* $\sigma : \mathrm{Spo}(B, B^+) \to \mathrm{Spec}(B)$ *denote the map* $x \mapsto supp(x)$ *attaching to every point of* $\mathrm{Spo}(B, B^+)$ *its support ideal. Notice that* $\sigma = supp \circ h$ *where* $supp : \mathrm{Spa}(B, B^+) \to \mathrm{Spec}(B)$ *also attaches the support ideal. Then* $Z = \sigma^{-1}(V(I))$ *for some prime ideal* $I \subseteq B$ *open for the topology in* B*.*

Proof. Any map $m^* : (B, B^+) \to (A, A)$ with A a discrete ring must factor through

$$(B/B^{\circ\circ}, B^+/B^{\circ\circ}),$$

by reducing to this case we may assume that B has the discrete topology. By proposition 1.2.34, $Z = h^{-1}(h(Z))$ and by corollary 1.2.15, Z is closed under vertical generization. Moreover, since Z is closed in $\mathrm{Spo}(B, B^+)$ it is also stable under vertical specialization. This implies that $Z = \sigma^{-1}(\sigma(Z))$. Indeed, any two points $x, y \in \mathrm{Spa}(B, B^+)$ with $supp(x) = supp(y) = \mathfrak{p}$ are vertical specializations of the trivial valuation on B with support $\mathfrak{p}$.

We only have left to prove that $\sigma(Z)$ is closed in the Zariski topology of $\mathrm{Spec}(B)$. Since B has the discrete topology, the support map admits a continuous section $triv : \mathrm{Spec}(B) \to \mathrm{Spa}(B, B^+)$ that assigns to a prime ideal $\mathfrak{p} \subseteq B$ the trivial valuation with support $\mathfrak{p}$. We have $\sigma(Z) = triv^{-1}(h(Z))$ so we may prove $h(Z)$ is closed instead. By proposition 1.2.34, $h(Z)$ is also closed under horizontal specialization, this gives that the complement of $h(Z)$ in $\mathrm{Spa}(B, B^+)$ is stable under (arbitrary) generization. $\mathrm{Spo}(B, B^+) \setminus Z = h^{-1}(\mathrm{Spa}(B, B^+) \setminus h(Z))$ by proposition 1.2.28 the set $\mathrm{Spa}(B, B^+) \setminus h(Z)$ is open as we needed to show. $\square$

1.3 The reduction functor

1.3.1 The v-topology for perfect schemes

This is the only section in which we will be forced to be set-theoretically careful, we advise the reader that does not wish the ignore the set-theoretic subtleties that arise in this section to review the definition and basic properties of cut-off cardinals that are given in [51] §4.

We will denote by $\mathrm{PCAlg}^{op}_{\mathbb{F}_p}$ the category of perfect affine schemes over $\mathbb{F}_p$. If κ is a cut-off cardinal we denote by $\mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa}$ the category of perfect affine schemes over $\mathbb{F}_p$ whose underlying topological space and whose ring of global sections have cardinality bounded by κ. Given $S = \mathrm{Spec}(A) \in \mathrm{PCAlg}^{op}_{\mathbb{F}_p}$ we associate to it a v-sheaf in Perf given by:

$$S^\diamondsuit((R, R^+)) = \{f : A \to R^+ | f \textit{ is a morphism of rings}\}$$

Remark 1.3.1. *Notice that* $\mathrm{Spec}(A)^\diamondsuit = \mathrm{Spd}(A, A)$ *when A is given the discrete topology. Later on, we will work with v-sheaves of the form* $\mathrm{Spd}(A, A)$ *where A can be given either the discrete topology or a more interesting topology and we might consider both kind of sheaves at the same time. To avoid having to specify the topology given to A every time, we will use* $\mathrm{Spec}(A)^\diamondsuit$ *whenever A is given the discrete topology and we will use* $\mathrm{Spd}(A, A)$ *when A is given a more interesting topology.*

Proposition 1.3.2. *If κ is a cut-off cardinal and $S \in \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa}$ then $S^\diamondsuit$ is a κ-small v-sheaf.*

Proof. Let $S = \mathrm{Spec}(A)$ then $\mathrm{Spa}(A((t^{\frac{1}{p^\infty}})), A[[t^{\frac{1}{p^\infty}}]])$ is a κ-small perfectoid space and the map

$$\mathrm{Spa}(A((t^{\frac{1}{p^\infty}})), A[[t^{\frac{1}{p^\infty}}]]) \to S^\diamondsuit$$

is surjective. $\square$

Proposition 1.3.2 gives rise to functors $\diamondsuit_\kappa : \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa} \to \widetilde{\mathrm{Perf}}_\kappa$ that are compatible when we vary κ and give rise to a functor $\diamondsuit : \mathrm{PCAlg}^{op}_{\mathbb{F}_p} \to \widetilde{\mathrm{Perf}}$

Proposition 1.3.3. *The functors* $\diamondsuit : \mathrm{PCAlg}^{op}_{\mathbb{F}_p} \to \widetilde{\mathrm{Perf}}$ *and* $\diamondsuit_\kappa : \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa} \to \widetilde{\mathrm{Perf}}_\kappa$ *are fully-faithful and commute with finite limits.*

Proof. This is a direct consequence of 1.2.32.

□

After embedding $\mathrm{PCAlg}^{op}_{\mathbb{F}_p}$ in $\widetilde{\mathrm{Perf}}$ one can define a Grothendieck topology on $\mathrm{PCAlg}^{op}_{\mathbb{F}_p}$ by considering a small family of maps of affine schemes, $(S_i \to T)_{i\in\mathcal{F}}$, to be a cover if the map $\coprod_{i\in\mathcal{F}} S_i^\diamondsuit \to T^\diamondsuit$ is a surjective map of v-sheaves. However, there is an intrinsic way of defining this topology which we now discuss.

Definition 1.3.4. *(See [5] 2.1)*

1. *A morphisms of qcqs schemes $S \to T$, is said to be universally subtrusive (or a v-cover) if for any valuation ring V and a map* $\mathrm{Spec}(V) \to T$ *there is an extension of valuation rings $V \subseteq W$ (see [56] 0ASG) and a map* $\mathrm{Spec}(W) \to S$ *making the following diagram commutative:*

$$\begin{tikzcd} \mathrm{Spec}(W) \arrow[r] \arrow[d] & S \arrow[d] \\ \mathrm{Spec}(V) \arrow[r] & T \end{tikzcd}$$

2. *A small family of morphisms in* $\mathrm{PCAlg}^{op}_{\mathbb{F}_p}$*, $(S_i \to T)_{i\in\mathcal{F}}$, is said to be universally subtrusive (or a v-cover) if there is a finite subset $\mathcal{F}' \subseteq \mathcal{F}$ for which $\coprod_{i\in\mathcal{F}'} S_i \to T$ is universally subtrusive.*

Lemma 1.3.5. *(See [5] 2.2) A morphism* $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *of affine schemes (not necessarily over $\mathbb{F}_p$) is universally subtrusive if and only if the map of topological spaces* $|f^{ad}| : |\mathrm{Spa}(B,B)| \to |\mathrm{Spa}(A,A)|$ *is surjective.*

Proof. Let $T = \mathrm{Spec}(A)$, $S = \mathrm{Spec}(B)$, $T^{ad} = \mathrm{Spa}(A,A)$ and $S^{ad} = \mathrm{Spa}(B,B)$. Assume f to be universally subtrusive and take $x \in |T^{ad}|$. Taking a representative we can consider x as a valuation $|\cdot|_x : A \to \Gamma_x$, which gives a valuation subring V of $Frac(A/supp(|\cdot|_x))$ together with a map $\mathrm{Spec}(V) \to \mathrm{Spec}(A)$. Since f is universally subtrusive we can take an extension of valuation rings W/V and a map $\mathrm{Spec}(W) \to \mathrm{Spec}(B)$ making diagram 1 above commutative. The map $B \to W$ induces a valuation $|\cdot|_y : B \to \Gamma_y$ and consequently a point $y \in S^{ad}$. Moreover, the composition $|\cdot|_{f(y)} : A \to B \to \Gamma_y$ is equivalent to $|\cdot|_x$ which proves that $|S^{ad}| \to |T^{ad}|$ is surjective. For the converse, given a map $\mathrm{Spec}(V) \to T$ we may consider the induced map $\mathrm{Spa}(K,V) \to T^{ad}$ with $K = Frac(V)$. The closed point of $\mathrm{Spa}(K,V)$ gives a point $x \in T^{ad}$ and by surjectivity of f^{ad} we may pick a point $y \in \mathrm{Spa}(B,B)$ lifting x. Consider the affinoid residue fields $(K_x, {K_x}^+)$ and $(K_y, {K_y}^+)$ at x and y respectively. We get the following commutative diagram:

$$\begin{array}{ccccc} & & \mathrm{Spec}(K_y^+) & \longrightarrow & S \\ & & \downarrow & & \downarrow \\ \mathrm{Spec}(V) & \longrightarrow & \mathrm{Spec}(K_x^+) & \longrightarrow & T \end{array}$$

Both K_y^+ and V are valuation extensions of K_x^+, consequently there is a valuation ring W extending both K_y^+ and V making the diagram commute (See [25] 1.1.14-f). This proves that f is universally subtrusive. $\square$

Lemma 1.3.6. *Let $f : S \to T$ be a morphism of perfect affine schemes over $\mathbb{F}_p$. The map $f^\diamondsuit : S^\diamondsuit \to T^\diamondsuit$ is a quasi-compact map of v-sheaves.*

Proof. By writing $B = A[t_i]_{i\in I}/JA[t_i]_{i\in I}$ for some variables t_i and an ideal J we can reduce to the cases where either f is a closed embedding or f is the base change of the structure map $g : \mathrm{Spec}(\mathbb{F}_p[t_i]_{i\in I}) \to \mathrm{Spec}(\mathbb{F}_p)$.

Let $X = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}$ it is enough to prove that $X \times_{T^\diamondsuit} S^\diamondsuit$ is quasi-compact for any map $\mathrm{Spa}(R, R^+) \to S^\diamondsuit$. For the later case, the basechange gives the sheaf $X \times \mathrm{Spec}(\mathbb{F}_p[t_i])^\diamondsuit$. This functor is represented by

$$\mathrm{Spa}(R\langle t_i^{\frac{1}{p^\infty}}\rangle_{i\in I}, R^+\langle t_i^{\frac{1}{p^\infty}}\rangle_{i\in I}),$$

which is affinoid perfectoid and consequently quasi-compact. For the former case let $B = A/J$, and let $Z = X \times_{T^\diamondsuit} S^\diamondsuit$. For a perfectoid Huber pair (L, L^+) we have:

$$Z(L, L^+) = \{r : (R, R^+) \to (L, L^+) \mid r(R \cdot J) = 0\}$$

This is the definition of a Zariski closed subset of X and by ([50] Lemma II.2.2) representable by an affinoid perfectoid space. In particular, Z is a quasi-compact v-sheaf. $\square$

Remark 1.3.7. *One can prove lemma 1.3.6 by observing that for a perfect discrete ring A we have the identity $\mathrm{Spd}(A, A)^\dagger = \mathrm{Spd}(A, A)$. Indeed, we can apply lemma 1.2.26.*

Proposition 1.3.8.
1. *Let $f : S \to T$ be a morphism of perfect affine schemes over $\mathbb{F}_p$. The map f is universally subtrusive if and only if $f^\diamondsuit : S^\diamondsuit \to T^\diamondsuit$ is a surjective map of v-sheaves.*
2. *A family of morphisms $(S_i \to T)_{i\in\mathcal{F}}$ is universally subtrusive if and only if $(\coprod_{i\in\mathcal{F}} S_i) \to T$ is a surjective map of v-sheaves.*

Proof. Let $T = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(B)$. Since the map of v-sheaves $f^\diamondsuit : S^\diamondsuit \to T^\diamondsuit$ is quasi-compact, by ([51] 12.11) it is a surjective map of v-sheaves if and only if $|f^\diamondsuit|$ is a surjective map of topological spaces. By proposition 1.2.12 and lemma 1.3.5, it suffices to prove that the map $\mathrm{Spo}(B, B) \to \mathrm{Spo}(A, A)$ is surjective if and only if the map $\mathrm{Spa}(B, B) \to \mathrm{Spa}(A, A)$ is. Functoriality and surjectivity of h proves one direction, and the converse direction is a direct consequence of 1.2.34.

For the second claim, it follows easily from above that a universally subtrusive family of maps $(S_i \to T)_{i\in\mathcal{F}}$ induces a surjective map of v-sheaves $(\coprod_{i\in\mathcal{F}} S_i) \to T$, actually a finite subfamily is already surjective. To prove the converse we have to take a family of maps $(S_i \to T)_{i\in\mathcal{F}}$ such that $(\coprod_{i\in\mathcal{F}} S_i^\diamondsuit) \to T^\diamondsuit$ is a surjective map of v-sheaves and prove there is a finite subset $\mathcal{F}' \subseteq \mathcal{F}$ for which $\coprod_{i\in\mathcal{F}'} S_i^\diamondsuit \to T^\diamondsuit$ is still surjective. Let $S_i = \mathrm{Spec}(R_i)$ and $T = \mathrm{Spec}(P)$ and consider affinoid perfectoid spaces $Y = \mathrm{Spa}(P((t^{\frac{1}{p^\infty}})), P[[t^{\frac{1}{p^\infty}}]])$ and $X_i = \mathrm{Spa}(R_i((t^{\frac{1}{p^\infty}})), R_i[[t^{\frac{1}{p^\infty}}]])$. The map $(\coprod_{i\in\mathcal{F}} X_i) \to Y$ is surjective and since Y is quasi-compact there is a finite subset $\mathcal{F}' \subseteq \mathcal{F}$ such that $(\coprod_{i\in\mathcal{F}'} X_i) \to Y$ is still surjective. An easy argument proves that for $\mathcal{F}'$ chosen in this way $(\coprod_{i\in\mathcal{F}'} S_i^\diamondsuit) \to T^\diamondsuit$ is also surjective. □

Remark 1.3.9. *In this context, one can discuss the analogue of example 1.1.4. Given an index set I and $\{V_i\}_{i\in I}$ a family of perfect valuation rings over $\mathbb{F}_p$, we construct the ring $R = \prod_{i\in I} V_i$. We call the affine schemes constructed in this way a scheme-theoretic product of points. They form a basis for the v-topology on $\mathrm{PCAlg}^{op}_{\mathbb{F}_p}$ (See [5] 6.2).*

Given a cut-off cardinal κ we let $\widetilde{\mathrm{SchPerf}}_\kappa$ be the topos associated to the site $\mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa}$ with the v-topology, and we will refer to an object in this topos as a κ-small scheme-theoretic v-sheaf. For any pair of cut-off cardinals $\kappa < \lambda$ we have a continuous fully-faithful embedding of sites $\iota^*_{\kappa,\lambda} : \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa} \to \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\lambda}$, which induces a morphism of topoi $\iota_{\kappa,\lambda} : \widetilde{\mathrm{SchPerf}}_\lambda \to \widetilde{\mathrm{SchPerf}}_\kappa$.

Proposition 1.3.10. *The functor $\iota^*_{\kappa,\lambda} : \widetilde{\mathrm{SchPerf}}_\kappa \to \widetilde{\mathrm{SchPerf}}_\lambda$ is fully-faithful (See [51] 8.2).*

Proof. It is enough to prove that the adjunction $\mathcal{F} \to \iota_{\kappa,\lambda,*}\iota^*_{\kappa,\lambda}\mathcal{F}$ is an isomorphism. Let

$$\mathcal{G} : \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\lambda} \to \mathrm{Sets}$$

be the presheaf with $S \mapsto \mathcal{G}(S)$ constructed as follows. Let $\mathcal{C}^\kappa_S$ denote the category of maps of affine schemes $S \to T$ with $T \in \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa}$. This category is cofiltered and there is a λ-small set of objects $I^\kappa_S \subseteq \mathcal{C}^\kappa_S$, that is cofinal in $\mathcal{C}^\kappa_S$. We let $\mathcal{G}(S) = \varinjlim_{T\in I^\kappa_S} \mathcal{F}(T)$, for any choice of I^κ_S. Unraveling the definitions we see that $\iota^*_{\kappa,\lambda}\mathcal{F}$ is the sheafification of $\mathcal{G}$.

We claim that $\mathcal{G}$ is already a sheaf. Indeed, since filtered colimits are exact it is enough to prove that any v-cover $S' \to S$ in $\mathrm{PCAlg}^{op}_{\mathbb{F}_p,\lambda}$ can be expressed as a filtered colimit of v-covers in $\mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa}$. Let $S = \mathrm{Spec}(A)$ and let $S' = \mathrm{Spec}(B)$, write $A = \varinjlim_{i\in I^\kappa_S} A_i$ and $B = \varinjlim_{j\in I^\kappa_{S'}} B_j$ with A_i and B_j κ-small rings, we may assume that the transition maps are all injective. By lemma 1.3.11 below we may assume that all morphisms $\mathrm{Spec}(A) \to \mathrm{Spec}(A_i)$ are v-covers. Consequently, the composition $S' \to S \to \mathrm{Spec}(A_i)$ is also a v-cover and whenever $S' \to \mathrm{Spec}(A_i)$ factors through a map $\mathrm{Spec}(B_j) \to \mathrm{Spec}(A_i)$ this later one is also a v-cover. We can replace our index sets I^κ_S and $I^\kappa_{S'}$ by a common index set I and replace the rings B_j by the smallest subring of B containing B_j and A_i for some $i \in I^\kappa_S$ so that we get a family indexed by I for which $(\mathrm{Spec}(B_i) \to \mathrm{Spec}(A_i))_{i\in I}$ is always defined and is a v-cover.

We get our desired expression

$$(S' \to S) = \varprojlim_{i\in I} (\mathrm{Spec}(B_i) \to \mathrm{Spec}(A_i))_{i\in I}.$$

Once we know $\iota^*_{\kappa,\lambda}\mathcal{F} = \mathcal{G}$, we compute $\iota_{\kappa,\lambda,*}\iota^*_{\kappa,\lambda}\mathcal{F}(S)$ for $S \in \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa}$ to be $\varinjlim_{T\in I^\kappa_S} \mathcal{F}(T)$, but since S is κ-small the identity map is cofinal in $\mathcal{C}^\kappa_S$ and $\varinjlim_{I^\kappa_S} \mathcal{F}(S) = \mathcal{F}(S)$ as we needed to show. $\square$

Lemma 1.3.11. *Let κ be a cut-off cardinal, $S \in \mathrm{PCAlg}^{op}_{\mathbb{F}_p}$ and $T \in \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa}$. Given a morphism $g : S \to T$, there is $T' \in \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa}$ together with morphisms $f : S \to T'$ and $h : T' \to T$ such that f is a v-cover and $g = h \circ f$.*

Proof. This lemma is purely of set-theoretic nature and contentless otherwise. Indeed, if S was κ-small we could simply choose $T' = S$ and f to be the identity. Lets treat the general case, let $S = \mathrm{Spec}(B)$ and $T = \mathrm{Spec}(A)$. By replacing A by its image in B we may assume $g^* : A \to B$ to be injective. We construct a countable sequence of subrings

$$A = A_0 \subseteq \dots \subseteq A_n \subseteq A_{n+1} \subseteq \dots B$$

with the property that each A_i is κ-small and that the image of the map $\mathrm{Spa}(B, B) \to \mathrm{Spa}(A_n, A_n)$ coincides with the image of $\mathrm{Spa}(A_{n+1}, A_{n+1}) \to \mathrm{Spa}(A_n, A_n)$. We do this inductively as follows: Assume A_n to be defined and let $Z_n \subseteq \mathrm{Spa}(A_n, A_n)$ be the image of $\mathrm{Spa}(B, B)$ in $\mathrm{Spa}(A_n, A_n)$. If x is an element of $\mathrm{Spa}(A_n, A_n)\setminus Z_n$ the valuation $|\cdot|_x : A_n \to \Gamma_x$ can't be extended to a valuation $|\cdot| : B \to \Gamma$. A compactness argument proves there are finitely many elements $\{a_1, \dots a_m\}$ such that $|\cdot|_x$ does not extend to $A_n[a_1, \dots, a_m] \subseteq B$. Since $\mathrm{Spa}(A_n, A_n) \setminus Z_n$ is κ-small, there is $\lambda < \kappa$ and a set $\{a_i\}_{i\in\lambda} \subseteq B$ such that $A_n[a_i]_{i\in\lambda}$ does not extend any $x \in \mathrm{Spa}(A_n, A_n) \setminus Z_n$. We let $A_{n+1} = A_n[a_i^{\frac{1}{p^\infty}}]_{i\in\lambda}$, clearly A_{n+1} satisfies the desired properties.

Let $A_\infty = \varinjlim_{i\in\mathbb{N}} A_i$, we claim that the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A_\infty)$ is a v-cover and that A_∞ is κ-small. Indeed, since each A_i is κ-small and since the cofinality of κ is larger than ω (See [51] 4.1) A_∞ is also κ-small. To prove it is a v-cover, we can use lemma 1.3.5 to prove instead that $\mathrm{Spa}(B, B) \to \mathrm{Spa}(A_\infty, A_\infty)$ is surjective. One verifies that $\mathrm{Spa}(A_\infty, A_\infty) = \varprojlim_{i\in\mathbb{N}} \mathrm{Spa}(A_i, A_i)$ as topological spaces. Given a compatible sequence $x_i \in \mathrm{Spa}(A_i, A_i)$ we define M_i to be the preimage of x_i in $\mathrm{Spa}(B, B)$. This gives a sequence of sets

$$\mathrm{Spa}(B, B) \supseteq M_0 \supseteq M_1 \dots M_n \supseteq \dots$$

Since the maps $\mathrm{Spa}(B, B) \to \mathrm{Spa}(A_i, A_i)$ are spectral maps of spectral topological spaces, each of the M_i is closed and compact in the patch topology and their intersection is non-empty. Any element in this intersection will map to the element $x_\infty \in \mathrm{Spa}(A_\infty, A_\infty)$ represented by the compatible sequence x_i. $\square$

We define $\widetilde{\mathrm{SchPerf}}$ as the big colimit $\bigcup_\kappa \widetilde{\mathrm{SchPerf}}_\kappa$ along all cut-off cardinals and the fully-faithful embeddings $\iota^*_{\kappa,\lambda}$. Objects in $\widetilde{\mathrm{SchPerf}}$ are called small scheme-theoretic v-sheaves.

The general formalism of topoi, specifically ([3] IV 4.9.4), allows us to promote $\diamondsuit_\kappa : \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa} \to \widetilde{\mathrm{Perf}}_\kappa$ to a morphism of topoi $f_\kappa : \widetilde{\mathrm{Perf}}_\kappa \to \widetilde{\mathrm{SchPerf}}_\kappa$ for which $f^*_\kappa|_{\mathrm{PCAlg}^{op}_{Fp,\kappa}} = \diamondsuit_\kappa$. Indeed, proposition 1.3.3 shows that $\diamondsuit_\kappa$ is left-exact and proposition 1.3.8 gives us continuity of $\diamondsuit_\kappa$.

Proposition 1.3.12. *1. Given two cut-off cardinals $\kappa < \lambda$ we have a commutative diagram of morphism of topoi:*

$$\begin{tikzcd} \widetilde{\mathrm{Perf}}_\lambda \arrow[r, "f_\lambda"] \arrow[d, "\iota_{\kappa,\lambda}"] & \widetilde{\mathrm{SchPerf}}_\lambda \arrow[d, "\iota_{\kappa,\lambda}"] \\ \widetilde{\mathrm{Perf}}_\kappa \arrow[r, "f_\kappa"] & \widetilde{\mathrm{SchPerf}}_\kappa \end{tikzcd}$$

2. We also have that the natural morphism $\iota^*_{\kappa,\lambda} \circ f_{\kappa,*} \to f_{\lambda,*} \circ \iota^*_{\kappa,\lambda}$ is an isomorphism.

Proof. The commutativity of morphism of topoi follows formally from the similar commutativity of continuous functors:

$$\begin{tikzcd} \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa} \arrow[r, "\diamondsuit_\kappa"] \arrow[d, "\iota^*_{\kappa,\lambda}"] & \widetilde{\mathrm{Perf}}_\kappa \arrow[d, "\iota^*_{\kappa,\lambda}"] \\ \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\lambda} \arrow[r, "\diamondsuit_\lambda"] & \widetilde{\mathrm{Perf}}_\lambda \end{tikzcd}$$

For the second claim, given an element $S \in \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\lambda}$ we let I^κ_S be an index set category as in the proof of 1.3.10. If $S = \mathrm{Spec}(A)$ we let $X = \mathrm{Spa}(A((t^{\frac{1}{p^\infty}})), A[[t^{\frac{1}{p^\infty}}]])$ and $Y = X \times_{S^\diamondsuit} X$. In a similar way, for $T \in I^\kappa_S$ with $T = \mathrm{Spec}(B)$ we let $X_T = \mathrm{Spa}(B((t^{\frac{1}{p^\infty}})), B[[t^{\frac{1}{p^\infty}}]])$ and $Y_T = X_T \times_{T^\diamondsuit} X_T$. The family of perfectoid spaces $(X_T)_{T\in I^\kappa_S}$ ($(Y_T)_{T\in I^\kappa_S}$ respectively) is cofinal in the category $\mathcal{C}^\kappa_X$ of maps $X \to X'$ with X' a κ-small perfectoid space ($\mathcal{C}^\kappa_Y$ respectively).

We get the following chain of isomorphisms:

$$
\begin{aligned}
\iota^*_{\kappa,\lambda} f_{\kappa,*}\mathcal{F}(S) &= \varinjlim_{T\in I^\kappa_S} Hom_{\widetilde{\mathrm{SchPerf}_\kappa}}(h_T, f_{\kappa,*}\mathcal{F}) &(1.1)\\
&= \varinjlim_{T\in I^\kappa_S} Hom_{\widetilde{\mathrm{Perf}_\kappa}}(f^*_\kappa h_T, \mathcal{F}) &(1.2)\\
&= \varinjlim_{T\in I^\kappa_S} Hom_{\widetilde{\mathrm{Perf}_\kappa}}(T^{\diamondsuit_\kappa}, \mathcal{F}) &(1.3)\\
&= \varinjlim_{T\in I^\kappa_S} Eq_{\widetilde{\mathrm{Perf}_\kappa}}(Hom(X_T, \mathcal{F}) \rightrightarrows Hom(Y_T, \mathcal{F})) &(1.4)\\
&= Eq_{\widetilde{\mathrm{Perf}_\lambda}}(\varinjlim_{T\in I^\kappa_S} Hom(X_T, \mathcal{F}) \rightrightarrows \varinjlim_{T\in I^\kappa_S} Hom(Y_T, \mathcal{F})) &(1.5)\\
&= Eq_{\widetilde{\mathrm{Perf}_\lambda}}(Hom(X_S, \iota^*_{\kappa,\lambda}\mathcal{F}) \rightrightarrows Hom(Y_S, \iota^*_{\kappa,\lambda}\mathcal{F})) &(1.6)\\
&= Hom_{\widetilde{\mathrm{Perf}_\lambda}}(S^{\diamondsuit_\lambda}, \iota^*_{\kappa,\lambda}\mathcal{F}) &(1.7)\\
&= Hom_{\widetilde{\mathrm{SchPerf}_\lambda}}(h_S, f_{\lambda,*}\iota^*_{\kappa,\lambda}\mathcal{F}) &(1.8)\\
&= f_{\lambda,*}\iota^*_{\kappa,\lambda}\mathcal{F}(S) &(1.9)
\end{aligned}
$$

□

Recall that a morphism of topoi consists of a pair of adjoint functors (f^*, f_*) such that f^* commutes with finite limits. By proposition 1.3.12 above we can gather all of the morphisms of topoi $f_\kappa : \widetilde{\mathrm{Perf}_\kappa} \to \widetilde{\mathrm{SchPerf}_\kappa}$ into a pair of adjoint functors $(f^*, f_*) : \widetilde{\mathrm{Perf}} \to \widetilde{\mathrm{SchPerf}}$ such that f^* commutes with finite limits. This is not a morphism of topoi because $\widetilde{\mathrm{Perf}}$ and $\widetilde{\mathrm{SchPerf}}$ are not topoi, but they behave as such.

Definition 1.3.13. *Let (f^*, f_*) the pair of adjoint functors described above, given $\mathcal{F} \in \widetilde{\mathrm{SchPerf}}$ we will denote $f^*\mathcal{F}$ by $\mathcal{F}^\diamondsuit$ and given $\mathcal{G} \in \widetilde{\mathrm{Perf}}$ we will denote $f_*\mathcal{G}$ by $(\mathcal{G})^{\mathrm{red}}$. We refer to $(-)^{\mathrm{red}}$ as the reduction functor.*

Remark 1.3.14. *The functor $(-)^{\mathrm{red}}$ will be very important for our purposes. To make this functor explicit take a small v-sheaf $\mathcal{F} \in \widetilde{\mathrm{Perf}}$ and $S \in \mathrm{PCAlg}^{op}_{\mathbb{F}_p}$. Adjunction tells us that $\mathcal{F}^{\mathrm{red}}(S) = Hom_{\widetilde{\mathrm{Perf}}}(S^\diamondsuit, \mathcal{F})$. We could have defined the functor in this way without invoking the formalism of topoi, but it will be useful to know that the "reduction" of small v-sheaf is a small scheme-theoretic v-sheaf.*

We can endow any small scheme-theoretic v-sheaf with a topological space in a similar fashion to definition 1.1.11. Given $\mathfrak{S} \in \widetilde{\mathrm{SchPerf}}$ we let $|\mathfrak{S}|$ denote the set of equivalence classes of maps $\mathrm{Spec}(k) \to \mathfrak{S}$, where k is a perfect field over $\mathbb{F}_p$. Two maps p_1, p_2 are equivalent if we can complete a commutative diagram as below:

$$\begin{tikzcd}
 & \mathrm{Spec}(k_1) \arrow[dr, "p_1"] & \\
\mathrm{Spec}(k_3) \arrow[ur, "q_1"] \arrow[rr, "p_3"] \arrow[dr, "q_2"'] & & \mathfrak{S} \\
 & \mathrm{Spec}(k_2) \arrow[ur, "p_2"'] &
\end{tikzcd}$$

Proposition 1.3.15. *Let $\mathfrak{S} \in \widetilde{\mathrm{SchPerf}}$ the following hold:*

1. *There is a pair of cut-off cardinals $\kappa < \lambda$ and a λ-small family $\{S_i\}_{i\in I}$ of objects in $\mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa}$ together with a surjective map $X = (\coprod_{i\in I} S_i) \to \mathfrak{S}$.*
2. *The small scheme-theoretic v-sheaf $R = X \times_{\mathfrak{S}} X$ has a similar cover $Y = (\coprod_{j\in J} T_j) \to R$, there is a natural map $|X| \to |\mathfrak{S}|$ which induces a bijection $|\mathfrak{S}| \cong |X|/|Y|$. We endow $|\mathfrak{S}|$ with the quotient topology induced by this bijection.*
3. *The topology on $|\mathfrak{S}|$ does not depend on the choices of X or Y.*
4. *Any map of small v-sheaves $\mathfrak{S}_1 \to \mathfrak{S}_2$ induces a continuous map of topological spaces $|\mathfrak{S}_1| \to |\mathfrak{S}_2|$.*

Proof. By definition $\mathfrak{S} \in \widetilde{\mathrm{SchPerf}}_\kappa$ for some cut-off cardinal κ, the category $\mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa}$ is a small category. By cofinality of cut-off cardinals we may pick λ larger than

$$\sup_{T\in \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa}} \mathfrak{S}(T).$$

We let $X = (\coprod_{T\in \mathrm{PCAlg}^{op}_{\mathbb{F}_p,\kappa}} \coprod_{x\in\mathfrak{S}(T)} T)$ with the evident projection map to $\mathfrak{S}$. We claim this map is surjective. This map is defined in $\widetilde{\mathrm{SchPerf}}_\lambda$ and it is enough to prove surjectivity there. Given $S \in \mathrm{PCAlg}_{\mathbb{F}_p,\lambda}$, we have $\mathfrak{S}(S) = \varinjlim_{T\in I^\kappa_S} \mathfrak{S}(T)$. Since this colimit is filtered, for a fixed map $g : S \to \mathfrak{S}$ we can find $(f : S \to T) \in I^\kappa_S$ and a map $h : T \to \mathfrak{S}$ with $g = h \circ f$. In particular, g factors through a map to X since X contains a copy of T mapping to $\mathfrak{S}$ via h.

We move on to the second claim. Given $x \in |X|$ we take the residue field inclusion $\iota_x : \mathrm{Spec}(k_x) \to X$. The composition $\mathrm{Spec}(k_x) \to X \to \mathfrak{S}$ defines an element of $|\mathfrak{S}|$. Suppose now that $x_1, x_2 \in |X|$, we must show that $(x_1, x_2) \in |X| \times |X|$ is in the image of $|Y|$ if and only if x_1 and x_2 define the same element in $|\mathfrak{S}|$. If the maps ι_{x_1} and ι_{x_2} are equivalent we get a map $\iota_{x_3} : \mathrm{Spec}(k_3) \to X \times_{\mathfrak{S}} X$. By replacing k_3 by a larger field if necessary, we may assume that ι_{x_3} lifts to Y and defines an element $y \in |Y|$. We see that y maps to (x_1, x_2) in $|X| \times |X|$. On the other hand, if there is $y \in Y$ mapping to (x_1, x_2)

and $\iota_y : \mathrm{Spec}(k_y) \to Y$ is the residue field map, the compositions $\mathrm{Spec}(k_y) \to Y \xrightarrow{\pi_i} X$ factor through $\iota_{x_i} : \mathrm{Spec}(k_{x_i}) \to X$. This proves that x_1 and x_2 map to the same point $|\mathfrak{S}|$.

For the third claim suppose we are given two covers $X_i \to \mathfrak{S}$ with $i \in \{1, 2\}$, we must show that the two quotient topologies coming from the surjections $|X_i| \to |\mathfrak{S}|$ agree. The small scheme theoretic v-sheaf $R = X_1 \times_{\mathfrak{S}} X_2$ admits a v-cover $X_3 \to R$ by the first claim. By replacing X_2 by X_3 we may assume that we have a commutative diagram of surjective maps:

$$\begin{tikzcd} X_2 \arrow[d] \arrow[dr] & \\ X_1 \arrow[r] & \mathfrak{S} \end{tikzcd}$$

Since $X_2 \to X_1$ is a v-cover we get a quotient map of topological spaces $|X_2| \to |X_1|$. If we give $|\mathfrak{S}|$ the quotient topology coming from the surjection $|X_1| \to |\mathfrak{S}|$ the composition map $|X_2| \to |\mathfrak{S}|$ is also a quotient map. This implies that the two topologies agree.

For the last claim, we may find covers $\mathfrak{S}_1$ and $\mathfrak{S}_2$ by X_1 and X_2 respectively forming the following commutative diagram:

$$\begin{array}{ccc} |\, X_2 \,| & \xrightarrow{q} & |\, \mathfrak{S}_2 \,| \\ \downarrow & & \downarrow \\ |\, X_1 \,| & \xrightarrow{q} & |\, \mathfrak{S}_1 \,| \end{array}$$

Both horizontal maps are quotient maps and the leftmost vertical map is continuous since it is induced by a morphism of unions of affine schemes, this prove the required continuity. $\square$

1.3.2 Reduction functor and formal adicness

Definition 1.3.16. *We say that a small scheme-theoretic v-sheaf $\mathcal{F}$ is reduced if the adjunction morphism $\mathcal{F} \to (\mathcal{F}^\lozenge)^{\mathrm{red}}$ is an isomorphism in $\widetilde{\mathrm{SchPerf}}$.*

We have the following formal consequences of our definition.

Proposition 1.3.17. *1. If S is a perfect scheme in characteristic p then the Yoneda functor h_S is reduced. (See [53] 18.3.1)*

2. The functor $\lozenge : \widetilde{\mathrm{SchPerf}} \to \widetilde{\mathrm{Perf}}$ is fully-faithful when restricted to small reduced v-sheaves.

Proof. The first claim follows from theorem 1.2.32. The second claim follows from adjunction. Indeed, if $\mathcal{F}$ is reduced and $\mathcal{G} \in \widetilde{\mathrm{SchPerf}}$ then:

$$Hom_{\widetilde{\mathrm{Perf}}}(\mathcal{G}^\lozenge, \mathcal{F}^\lozenge) = Hom_{\widetilde{\mathrm{SchPerf}}}(\mathcal{G}, (\mathcal{F}^\lozenge)^{\mathrm{red}}) = Hom_{\widetilde{\mathrm{SchPerf}}}(\mathcal{G}, \mathcal{F})$$

$\square$

Question 1.3.18. *Are perfect algebraic spaces reduced? If this was the case much of the formalism of specialization goes through in this generality.*

Intuitively speaking, the reduction functor kills all topological nilpotents and removes analytic points from our v-sheaf. Below, we try to justify why one can think of this reduction functor as an analogue of taking the underlying reduced subscheme of a formal scheme.

Lemma 1.3.19. *The scheme-theoretic v-sheaf $(\mathbb{Z}_p{}^\diamondsuit)^{\mathrm{red}}$ is represented by $\mathbb{F}_p$.*

Proof. This is a direct consequence of lemma 1.2.30.

$\square$

For a Huber pair (A, A^+) over $\mathbb{Z}_p$, we let A_{red} denote the perfection of $A/(A \cdot A^{\circ\circ})$ where $A \cdot A^{\circ\circ}$ is the ideal generated by the set of topologically nilpotent elements. The following statement generalizes lemma 1.3.19

Proposition 1.3.20. *Let X be a pre-adic space over $\mathbb{Z}_p$ and let X^{na} be the reduced adic space associated to the non-analytic locus of proposition 1.1.28. The following hold:*

1. *The map $(X^{na,\diamondsuit})^{\mathrm{red}} \to (X^\diamondsuit)^{\mathrm{red}}$ is an isomorphism.*
2. *If $X = \mathrm{Spa}(A, A^+)$ for (A, A^+) a Huber pair over $\mathbb{Z}_p$, then $\mathrm{Spd}(A, A^+)^{\mathrm{red}}$ is represented by $\mathrm{Spec}(A_{\mathrm{red}})$.*

Proof. By theorem 1.2.32 if $S = \mathrm{Spec}(R) \in \mathrm{PCAlg}^{op}_{\mathbb{F}_p}$ then morphisms $S^\diamondsuit \to X$ are given by maps of pre-adic spaces $f : \mathrm{Spa}(R, R) \to X$, and they must factor through the non-analytic locus which proves the first claim.

The non-analytic locus of $\mathrm{Spa}(A, A^+)$ is represented by the Huber pair $(A/A^{\circ\circ}\cdot A, A^{\circ\circ}\cdot A^+)$. Since R is perfect the map $f^* : A/A \cdot A^{\circ\circ} \to R$ factors uniquely through its perfection.

$\square$

We now justify the intuition behind thinking of diamonds as purely analytic objects.

Proposition 1.3.21. *For a quasi-separated diamond Y the associated reduced functor Y^{red} is the empty-sheaf.*

Proof. We need to prove that for a perfect scheme S there are no morphisms $S^\diamondsuit \to Y$. It is enough to prove this for $S = \mathrm{Spec}(k)$ the spectrum of an algebraically closed field. Suppose there is such a map $f : S^\diamondsuit \to Y$, and let $y \in |Y|$ be the unique point in the image of $|f|$. We consider Y_y the sub-v-sheaf of points of $\mathrm{Spa}(R, R^+) \to Y$ for which $|\mathrm{Spa}(R, R^+)| \to |Y|$ factors through y. The map f factors through Y_y and by ([51] 11.10) it is a quasi-separated diamond with $|Y_y|$ consisting of one point. We can use ([51] 21.9) to write $Y_y = \mathrm{Spa}(C, O_C)/\underline{G}$ with C a non-Archimedean algebraically closed field in characteristic p and $\underline{G}$ a profinite group acting continuously and faithfully on C.

Consider the v-cover $S' = \mathrm{Spa}(K_1, O_{K_1}) \to \mathrm{Spec}(k)^\diamondsuit$ where K_1 is an algebraic closure of $k((t^{\frac{1}{p^\infty}}))$. Similarly, let $T = \mathrm{Spa}(K_2, O_{K_2})$ where K_2 is an algebraically closed non-Archimedean field containing k discretely and whose value group $\Gamma_{K_2} \subseteq \mathbb{R}^{\geq 0}$ has at least two elements that are linearly independent when we treat $\Gamma_{K_2} \setminus \{0\}$ as vector space over $\mathbb{Q}$. By our hypothesis on K_2, we can find two continuous embeddings $\iota_i^* : K_1 \to K_2$ such that $|\iota_1^*(K_1)|_{\Gamma_{K_2}} \cap |\iota_2^*(K_1)|_{\Gamma_{K_2}} = 1$ and in particular, such that $\iota_1^*(K_1) \cap \iota_2^*(K_1) = k$.

The composition of $f : S^\diamondsuit \to Y_y$ with the natural projection $\mathrm{Spa}(K_1, {K_1}^+) \to S^\diamondsuit$ gives a map $[g] : \mathrm{Spa}(K_1, {K_1}^+) \to Y_y$ such that $[g] \circ \iota_1 = [g] \circ \iota_2$. Since both $\mathrm{Spa}(K_1, {K_1}^+)$ and $\mathrm{Spa}(K_2, {K_2}^+)$ are algebraically closed fields the sets of maps to Y_y are given by G-orbits of maps to $\mathrm{Spa}(C, O_C)$, that is $Hom(\mathrm{Spa}(K_i, {K_i}^+), Y_y) = Hom(\mathrm{Spa}(K_i, {K_i}^+), \mathrm{Spa}(C, O_C))/G$. Let $g^* : (C, O_C) \to (K_1, O_{K_1})$ represent $[g]$ in $Hom(\mathrm{Spa}(K_1, {K_1}^+), Y_y)$, we get maps $\iota_i^* \circ g^* : (C, O_C) \to (K_2, O_{K_2})$ and since $[g] \circ \iota_1 = [g] \circ \iota_1$ we have $\iota_1^* \circ g^*(C) = \iota_2^* \circ g^*(C) \subseteq k$. This contradicts that k has the discrete topology and that C is a non-Archimedean field, the contradiction shows that the map $f : S^\diamondsuit \to Y_y$ does not exist.

□

Recall that a morphism of adic spaces $X \to Y$ is said to be adic if the image of an analytic point is again an analytic point. For v-sheaves we can define a related notion.

Definition 1.3.22. *We say that a morphism of v-sheaves $\mathcal{F} \to \mathcal{G}$ is formally adic if the commutative diagram that one obtains from adjunction:*

$$\begin{tikzcd}
(\mathcal{F}^{\mathrm{red}})^\diamondsuit \arrow[r] \arrow[d] & (\mathcal{G}^{\mathrm{red}})^\diamondsuit \arrow[d] \\
\mathcal{F} \arrow[r] & \mathcal{G}
\end{tikzcd}$$

is a Cartesian diagram.

We warn the reader that although the notion of a morphism of adic spaces to be adic is related to the morphism of v-sheaves being formally adic neither of this notions implies the other.

Example 1.3.23. *Take a perfect field k in characteristic p together with a rank 1 valuation subring $O_k \subseteq k$ with the discrete topology. The morphism of adic spaces* $\mathrm{Spa}(k, O_k) \to \mathrm{Spa}(\mathbb{F}_p, \mathbb{F}_p)$ *is adic. Nevertheless, the induced morphism* $\mathrm{Spd}(k, O_k) \to \mathrm{Spd}(\mathbb{F}_p, \mathbb{F}_p)$ *is not formally adic since* $\mathrm{Spd}(k, O_k)^{\mathrm{red}}$ *is represented by* $\mathrm{Spec}(k)$*. Observe that* $\mathrm{Spo}(k, O_k)$ *has a meromorphic point that is not bounded.*

Example 1.3.24. *Take a non-Archimedean perfect field K in characteristic p and consider the morphism $id : Spa(K_1, O_{K_1}) \to Spa(K_2, O_{K_2})$ where $K_2 = K$ given the discrete topology and $K_1 = K$ given the topology induced by the norm. This morphism is not adic, nevertheless the reduction diagram looks like this:*

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & \operatorname{Spec}(K_2)^\lozenge \arrow[d] \\
\operatorname{Spd}(K_1, O_{K_1}) \arrow[r] & \operatorname{Spd}(K_2, O_{K_2})
\end{tikzcd}$$

Which is Cartesian.

Although the notion of formal adicness does not recover the notion of adicness in general, it will in some important situations:

Proposition 1.3.25. *Let (A, A) and (B, B) be formal Huber pairs over $\mathbb{Z}_p$ with ideals of definition I_A and I_B respectively. A morphism of adic spaces $\operatorname{Spa}(A, A) \to \operatorname{Spa}(B, B)$ is adic if and only if the corresponding morphism of v-sheaves $\operatorname{Spd}(A, A) \to \operatorname{Spd}(B, B)$ is formally adic.*

Proof. The reduction diagram looks as follows:

$$\begin{tikzcd}
\operatorname{Spec}(A_{\mathrm{red}})^\lozenge \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& (\operatorname{Spec}(A/I_B)^{perf})^\lozenge \arrow[r] \arrow[d] & \operatorname{Spec}(B_{\mathrm{red}})^\lozenge \arrow[d] \\
& \operatorname{Spd}(A, A) \arrow[r] & \operatorname{Spd}(B, B)
\end{tikzcd}$$

Continuity of the morphism $B \to A$ ensures that $I_B^n \subseteq I_A$ for some n. In this context, the morphism is adic if and only if $I_B \cdot A$ is an ideal of definition of A which happens if and only if $I_A^m \subseteq A \cdot I_B$ for some m. If the morphism is adic, then A/I_A and $(A/A \cdot I_B)$ become isomorphic after taking perfection which gives formal adicness. Conversely, if the morphism is formally adic, by hypothesis the rings $(A/I_B)^{perf}$, and A_{red} are isomorphic with the isomorphism being induced by the natural surjective ring map with source $(A/p)^{perf}$. This implies that the ideals I_A and I_B define the same Zariski closed subset in $\operatorname{Spec}(A)$. In particular, the elements of I_A are nilpotent in A/I_B, and since I_A is finitely generated $I_A^m \subseteq I_B$ for some m. ☐

Proposition 1.3.26. *1. If $\mathcal{F} \to \mathcal{H}$ and $\mathcal{H} \to \mathcal{G}$ are formally adic, the composition $\mathcal{F} \to \mathcal{G}$ is formally adic.*

2. If $\mathcal{F} \to \mathcal{H}$ is formally adic, the basechange $\mathcal{G} \times_{\mathcal{H}} \mathcal{F} \to \mathcal{G}$ is formally adic.

Proof. The first claim is clear. For the second one we get a Cartesian diagram:

$$\begin{tikzcd}
(\mathcal{G}^{\mathrm{red}})^\lozenge \times_{(\mathcal{H}^{\mathrm{red}})^\lozenge} (\mathcal{F}^{\mathrm{red}})^\lozenge \arrow[r] \arrow[d] & (\mathcal{F}^{\mathrm{red}})^\lozenge \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\
(\mathcal{G}^{\mathrm{red}})^\lozenge \arrow[r] & (\mathcal{H}^{\mathrm{red}})^\lozenge \arrow[r] & \mathcal{H}
\end{tikzcd}$$

The functors $(-)^{\mathrm{red}}$ and $(-)^{\diamondsuit}$ commute with finite limits. This gives $(\mathcal{G}^{\mathrm{red}})^{\diamondsuit} \times_{(\mathcal{H}^{\mathrm{red}})^{\diamondsuit}} (\mathcal{F}^{\mathrm{red}})^{\diamondsuit} = ((\mathcal{G} \times_{\mathcal{H}} \mathcal{F})^{\mathrm{red}})^{\diamondsuit}$, and proves that

$$\begin{tikzcd}
((\mathcal{G} \times_{\mathcal{H}} \mathcal{F})^{\mathrm{red}})^{\diamondsuit} \arrow[r] \arrow[d] & \mathcal{G} \times_{\mathcal{H}} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\
(\mathcal{G}^{\mathrm{red}})^{\diamondsuit} \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{H}
\end{tikzcd}$$

is also Cartesian. $\square$

Definition 1.3.27. *We say that a v-sheaf $\mathcal{F}$ over $\mathbb{Z}_p^{\diamondsuit}$ is formally p-adic if the morphism $\mathcal{F} \to \mathbb{Z}_p^{\diamondsuit}$ is formally adic.*

Over $\mathbb{Z}_p$ the situation of example 1.3.24 does not happen.

Proposition 1.3.28. *Suppose we have a Huber pair (A, A^+) and a map $f : \mathrm{Spa}(A, A^+) \to \mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$, if $f^{\diamondsuit}$ is formally adic then f is adic (as a morphism of adic spaces).*

Proof. Let $U \subseteq \mathrm{Spa}(A, A^+)$ the open subset of analytic points. It is easy to verify that this open embedding is formally adic because $\mathrm{Spec}(A_{\mathrm{red}})^{\diamondsuit} \to \mathrm{Spd}(A, A^+)$ factors through the complement of $U^{\diamondsuit}$ and because by proposition 1.3.21 $(U^{\diamondsuit})^{\mathrm{red}} = \emptyset$ holds. Since formal adicness is preserved by composition $U^{\diamondsuit} \to \mathbb{Z}_p^{\diamondsuit}$ is formally adic. By formal adicness the map $U^{\diamondsuit} \to \mathbb{Z}_p^{\diamondsuit}$ must factor through $\mathbb{Q}_p^{\diamondsuit}$. This proves $f(U) \subseteq \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ which proves that f is adic. $\square$

Recall that a v-sheaves $\mathcal{F}$ is said to be separated if the diagonal $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is a closed immersion (See [51] 10.7). We need the following related notion:

Definition 1.3.29.
1. *We say that a map of v-sheaves $\mathcal{F} \to \mathcal{G}$ is formally closed if it is a formally adic closed immersion.*
2. *We say that a v-sheaf is formally separated if the diagonal map $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is formally closed.*

Lemma 1.3.30. *The v-sheaf $\mathbb{Z}_p^{\diamondsuit}$ is formally separated.*

Proof. We need to prove that the diagonal $\mathbb{Z}_p^{\diamondsuit} \to \mathbb{Z}_p^{\diamondsuit} \times \mathbb{Z}_p^{\diamondsuit}$ is a closed immersion of perfectoid spaces after any basechange by maps $\mathrm{Spa}(R, R^+) \to \mathbb{Z}_p^{\diamondsuit} \times \mathbb{Z}_p^{\diamondsuit}$, with $\mathrm{Spa}(R, R^+) \in \mathrm{Perf}$. This amounts to proving that the locus on which two untilts agree is closed inside $|\mathrm{Spa}(R, R^+)|$ and representable by a perfectoid space. Now, each untilt is individually cut out of $\mathrm{Spa}(W(R^+), W(R^+)) \setminus \{V([\varpi])\}$ as a closed Cartier divisor (See [53] 11.3.1). We can take the intersection which will define a Zariski closed subset in each of the untilts, but Zariski closed subsets of a perfectoid space are representable by some other perfectoid space. The tilt of such a closed immersion represents this basechange.

To prove the diagonal is formally adic we compute directly $(\mathbb{Z}_p^{\diamondsuit} \times_{\mathbb{F}_p^{\diamondsuit}} \mathbb{Z}_p^{\diamondsuit})^{\mathrm{red}} = \mathbb{F}_p$ since $(-)^{\mathrm{red}}$ commutes with limits. The basechange $\mathbb{F}_p^{\diamondsuit} \times_{\mathbb{Z}_p^{\diamondsuit} \times_{\mathbb{F}_p^{\diamondsuit}} \mathbb{Z}_p^{\diamondsuit}} \mathbb{Z}_p^{\diamondsuit}$ agrees with $\mathbb{F}_p^{\diamondsuit}$. $\square$

Proposition 1.3.31. *If a v-sheaf $\mathcal{F}$ is formally p-adic, then the diagonal $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is formally adic.*

Proof. We have a formally adic map $\mathcal{F} \to \mathbb{Z}_p^\diamondsuit$, and since formal adicness is preserved by basechange and composition we get a formally adic map $\mathcal{F} \times_{\mathbb{Z}_p^\diamondsuit} \mathcal{F} \to \mathbb{Z}_p^\diamondsuit$. By the two out of three property of Cartesian diagrams, the diagonal map $\mathcal{F} \to \mathcal{F} \times_{\mathbb{Z}_p^\diamondsuit} \mathcal{F}$ is also formally adic. Now, $\mathcal{F} \times_{\mathbb{Z}_p^\diamondsuit} \mathcal{F}$ is the basechange of the diagonal map $\mathbb{Z}_p^\diamondsuit \to \mathbb{Z}_p^\diamondsuit \times \mathbb{Z}_p^\diamondsuit$ by the projection map $\mathcal{F} \times \mathcal{F} \to \mathbb{Z}_p^\diamondsuit \times \mathbb{Z}_p^\diamondsuit$. This gives us that $\mathcal{F} \times_{\mathbb{Z}_p^\diamondsuit} \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is also formally adic. Since formal adicness is preserved by composition, $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is also formally adic as we needed to show. $\square$

Lemma 1.3.32. *The diagonal $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is formally adic if and only if the adjunction morphism $(\mathcal{F}^{\mathrm{red}})^\diamondsuit \to \mathcal{F}$ is injective. In this case, if (A, A^+) is a perfectoid Huber pair, and $m \in \mathcal{F}(A, A^+)$ then $m \in (\mathcal{F}^{\mathrm{red}})^\diamondsuit(A, A^+)$ if and only if* $\mathrm{Spa}(A, A^+)$ *admits a v-cover* $\mathrm{Spa}(R, R^+) \to \mathrm{Spa}(A, A^+)$ *and a morphism* $\mathrm{Spec}(R^+)^\diamondsuit \to \mathcal{F}$ *making the following diagram commutative:*

$$\begin{array}{ccc} \mathrm{Spa}(R, R^+) & \longrightarrow & \mathrm{Spec}(R^+)^\diamondsuit \\ \big\downarrow & & \big\downarrow \\ \mathrm{Spa}(A, A^+) & \xrightarrow{\quad m \quad} & \mathcal{F} \end{array}$$

Proof. In general, a map of sheaves $\mathcal{G} \to \mathcal{F}$ is injective if and only if $(\mathcal{G} \times \mathcal{G}) \times_{\mathcal{F} \times \mathcal{F}} \mathcal{F} = \mathcal{G}$. We can apply this reasoning to the map $(\mathcal{F}^{\mathrm{red}})^\diamondsuit \to \mathcal{F}$.

For the second claim let $\mathcal{C}_R$ be the category of maps $\mathrm{Spa}(R, R^+) \to S^\diamondsuit$ with $S \in \mathrm{PCAlg}_{\mathbb{F}_p}^{op}$, this category is cofiltered. Now, $(\mathcal{F}^{\mathrm{red}})^\diamondsuit$ is the sheafification of the functor that assigns to (R, R^+):

$$\varinjlim_{S^\diamondsuit \in \mathcal{C}_R} Hom(S^\diamondsuit, \mathcal{F}).$$

But the evident map $\mathrm{Spa}(R, R^+) \to \mathrm{Spec}(R^+)^\diamondsuit$ is cofinal in $\mathcal{C}_R$. That is, $(\mathcal{F}^{\mathrm{red}})^\diamondsuit$ is the sheafification of the presheaf that assigns $(R, R^+) \mapsto Hom(\mathrm{Spec}(R^+)^\diamondsuit, \mathcal{F})$. The description given in the statement above is what one gets from taking sheafification and assuming injectivity of $(\mathcal{F}^{\mathrm{red}})^\diamondsuit \to \mathcal{F}$. $\square$

The following lemma will be key for our theory of specialization, it roughly says that formally adic closed immersions behave as expected:

Lemma 1.3.33. *Let (A, A^+) be a perfectoid Huber pair and let $\mathcal{F} \to \mathrm{Spd}(A^+, A^+)$ be formally adic closed immersion. Then $(\mathcal{F}^{\mathrm{red}})^\diamondsuit = \mathrm{Spec}(A^+/J)^\diamondsuit$ for some open ideal $J \subseteq A^+$.*

Proof. Since $\mathcal{F} \to \mathrm{Spd}(A^+, A^+)$ is a closed immersion, $|\mathcal{F}| \subseteq \mathrm{Spo}(A^+, A^+)$ is a closed subset and we have a Cartesian diagram,

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d] & \mathrm{Spd}(A^+, A^+) \arrow[d] \\
\underline{|\,\mathcal{F}\,|} \arrow[r] & \underline{\mathrm{Spo}(A^+, A^+)}
\end{tikzcd}$$

By proposition 1.3.20, $(\mathrm{Spd}(A^+, A^+)^{\mathrm{red}})^\diamond = \mathrm{Spec}(A^+_{\mathrm{red}})^\diamond$ which is also a closed subsheaf of $\mathrm{Spd}(A^+, A^+)$. By formal adicness $(\mathcal{F}^{\mathrm{red}})^\diamond$ is a closed subsheaf of $\mathrm{Spd}(A^+, A^+)$ given again by the topological condition $|\mathcal{F}| \cap |\mathrm{Spec}(A^+_{\mathrm{red}})|$. By lemma 1.3.32 a map $\mathrm{Spa}(R, R^+) \to \mathcal{F}$ factors through $(\mathcal{F}^{\mathrm{red}})^\diamond$ if after possibly replacing R by a v-cover it factors through $\mathrm{Spec}(R^+)^\diamond \to \mathcal{F} \cap \mathrm{Spec}(A^+_{\mathrm{red}})^\diamond$. This proves that $|(\mathcal{F}^{\mathrm{red}})^\diamond|$ is a schematic closed subset of $\mathrm{Spo}(A^+, A^+)$ as in definition 1.2.35. By proposition 1.2.36 it is a Zariski closed subset corresponding to an open ideal $J \subseteq A^+$, this proves the claim. $\square$

Example 1.3.34. *Let Z denote the complement $\mathrm{Spd}(\mathbb{F}_p[t], \mathbb{F}_p[t]) \setminus N_{t<<1}$ of the analytic localization $N_{t<<1}$. Then $Z \to \mathrm{Spd}(\mathbb{F}_p[t], \mathbb{F}_p[t])$ is a closed immersion that is not formally closed. One may glue two copies of $\mathrm{Spd}(\mathbb{F}_p[t], \mathbb{F}_p[t])$ along Z to get a v-sheaf Y that is separated but not formally separated. Y^{red} is represented by the affine line with two origins but the adjunction map $(Y^{\mathrm{red}})^\diamond \to Y$ is not injective.*

We will often use implicitly the following easy result.

Lemma 1.3.35. *Let $\mathcal{F}$ and $\mathcal{G}$ be two small v-sheaves, and $f : \mathcal{F} \to \mathcal{G}$ a map between them. Suppose that $\mathcal{F} \times_{\mathcal{G}} (\mathcal{G}^{red})^\diamond$ is representable by a reduced scheme-theoretic v-sheaf (definition 1.3.16) and that $\mathcal{G}$ is formally separated, then f is formally adic.*

Proof. Let T be a reduced scheme-theoretic v-sheaf such that $T^\diamond = \mathcal{F} \times_{\mathcal{G}} (\mathcal{G}^{red})^\diamond$. Since T is reduced we have $(T^\diamond)^{\mathrm{red}} = T$ and consequently $((T^\diamond)^{\mathrm{red}})^\diamond = T^\diamond$. Recall that for any pair of adjoint functors (L, R) the compositions $R \to R \circ L \circ R \to R$ and $L \to L \circ R \circ L \to L$ are the identity. Since $\mathcal{G}$ is formally separated the adjunction map $(\mathcal{G}^{\mathrm{red}})^\diamond \to \mathcal{G}$ is injective. Since $(-)^{\mathrm{red}}$ is a right adjoint the map $f : ((\mathcal{G}^{\mathrm{red}})^\diamond)^{\mathrm{red}} \to \mathcal{G}^{\mathrm{red}}$ is also injective. Now, the map f is injective and the identity of $\mathcal{G}^{\mathrm{red}}$ factors through it, this implies that f is an isomorphism. We can compute:

$$\begin{aligned}
(T^\diamond)^{\mathrm{red}} &= (\mathcal{F} \times_{\mathcal{G}} (\mathcal{G}^{\mathrm{red}})^\diamond)^{\mathrm{red}} \\
&= \mathcal{F}^{\mathrm{red}} \times_{\mathcal{G}^{\mathrm{red}}} ((\mathcal{G}^{\mathrm{red}})^\diamond)^{\mathrm{red}} \\
&= \mathcal{F}^{\mathrm{red}} \times_{\mathcal{G}^{\mathrm{red}}} \mathcal{G}^{\mathrm{red}} \\
&= \mathcal{F}^{\mathrm{red}}
\end{aligned}
\qquad
\begin{aligned}
(\mathcal{F}^{\mathrm{red}})^\diamond &= ((T^\diamond)^{\mathrm{red}})^\diamond \\
&= T^\diamond
\end{aligned}$$

$\square$

1.4 Specialization

In this section we review the specialization map in the context of formal schemes and generalize it to the context of v-sheaves. We identify a class of v-sheaves, that we call kimberlites, whose specialization maps behave like those of formal schemes. We prove some abstract statement on the behavior of the specialization map in this context, which we will later use when we discuss the examples of interest.

1.4.1 Specialization for Tate Huber pairs

Definition 1.4.1. *Given a Tate Huber pair (A, A^+) over $\mathbb{Z}_p$ and a pseudo-uniformizer $\varpi \in A$, we define the specialization map $\mathrm{sp}_A : |\mathrm{Spa}(A, A^+)| \to |\mathrm{Spec}(A^+_{\mathrm{red}})|$ by sending a valuation $|\cdot|_x \in |\mathrm{Spa}(A, A^+)|$ to the ideal $\mathfrak{p} \subseteq A^+$ given by $\mathfrak{p} = \{a \in A^+ \mid |a|_x < 1\}$*

These maps of sets are functorial in the category of Tate Huber pairs.

Proposition 1.4.2. *(See [4] 8.1.2) The specialization map $\mathrm{sp}_A : |\mathrm{Spa}(A, A^+)| \to |\mathrm{Spec}(A^+_{\mathrm{red}})|$ is a continuous, surjective, spectral and closed map of spectral topological spaces.*

Strictly totally disconnected spaces form a basis for the pro-étale topology on Perf. In particular, any small v-sheaf admits a surjective map from a union of totally disconnected spaces. Moreover, as the following proposition shows, the specialization map for these spaces is usefully nice.

Proposition 1.4.3. *For a strictly totally disconnected space $\mathrm{Spa}(R, R^+)$, the specialization map sp_R is a homeomorphism.*

Proof. By proposition 1.4.2 the map is surjective and a quotient map so it is enough to prove injectivity. Suppose $x, y \in |\mathrm{Spa}(R, R^+)|$ map to the same point in $|\mathrm{Spec}(R^+_{\mathrm{red}})|$. We claim that x and y are in the same connected component of $|\mathrm{Spa}(R, R^+)|$. Indeed, let π_x and π_y be the connected components of x and y respectively. The closed-open subsets $U \subseteq \mathrm{Spa}(R, R^+)$ are Zariski closed subsets defined by an idempotent $1_U \in R^+$. The ones containing x are precisely those for which $|1_U|_x = 1$ or equivalently for which $1_U \notin \mathrm{sp}_R(x) \subseteq R^+$. By assumption $\mathrm{sp}_R(x) = \mathrm{sp}_R(y)$ so x and y are contained in the same closed-opens, this gives $\pi_x = \pi_y$.

By proposition 1.1.8, π_x is representable by $\mathrm{Spa}(C, C^+)$ for some perfectoid field C and open valuation subring C^+. By functoriality of the specialization map it is enough to prove that the maps sp_C and $|\mathrm{Spec}(C^+/\varpi)| \to |\mathrm{Spec}(R^+/\varpi)|$ are injective. The former is injective by lemma 1.4.4 below. To prove injectivity of the later map we argue as follows: $\pi_x = \bigcap U$ where U ranges over the closed-open subsets of $|\mathrm{Spa}(R, R^+)|$ containing x. Each closed-open $U \subseteq \mathrm{Spa}(R, R^+)$ is of the form $U = \mathrm{Spa}(R_U, {R_U}^+)$ and if U^c denotes the complement of U then $R^+ = R^+_U \times R^+_{U^c}$ as topological rings. In particular, the map $R^+ \to R^+_U$ is surjective. We have that C^+ is the ϖ-adic completion of $\varinjlim_{x \in U} R^+_U$ which implies that the image of $R^+ \to C^+$ is dense. Consequently, $\mathrm{Spec}(C^+/\varpi) \to \mathrm{Spec}(R^+/\varpi)$ is a closed immersion and injective. □

Lemma 1.4.4. *1. Given a non-Archimedean field K there is an order preserving bijection between open and bounded valuation subrings K^+ of K, and valuation subrings of $O_K/K^{\circ\circ}$, given by $K^+ \mapsto K^+/K^{\circ\circ}$*

2. *Given K as above and an open and bounded valuation subring K^+ the specialization map sp_K is a homeomorphism.*

Proof. This is well known and the proof is left to the reader.

□

Remark 1.4.5. *Although the construction of V in the proof above does not depend of the choice of $\varpi = (\varpi_i)_{i\in I}$, the ring V' very much depends of this choice. This is in agreement with remark 1.1.6.*

1.4.2 Specializing v-sheaves

We now discuss the specialization map for v-sheaves. The idea is to descend the specialization map from the case of formal Huber pairs.

Definition 1.4.6. *We say that a small v-sheaf $\mathcal{F}$ is v-locally formal if there is a set I, a family $(B_i, B_i)_{i\in I}$ of formal Huber pairs over $\mathbb{Z}_p$ and a surjective map of v-sheaves*

$$\coprod_{i\in I} \mathrm{Spd}(B_i, B_i) \to \mathcal{F}.$$

Definition 1.4.7. *Let $\mathcal{F}$ be a small v-sheaf, $\mathrm{Spa}(A, A^+)$ an affinoid perfectoid space in characteristic p and $f : \mathrm{Spa}(A, A^+) \to \mathcal{F}$ a map of v-sheaves.*

1. *We say that $\mathcal{F}$ formalizes f (or that f is formalizable) if there exists a commutative diagram as follows:*

$$\begin{tikzcd} \mathrm{Spa}(A, A^+) \arrow[r] \arrow[d, "\iota"] & \mathcal{F} \\ \mathrm{Spd}(A^+, A^+) \arrow[ur] & \end{tikzcd}$$

2. *We say that $\mathcal{F}$ v-formalizes f if there is a v-cover $g : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ of affinoid perfectoid spaces for which $\mathcal{F}$ formalizes $f \circ g$.*

3. *We say that $\mathcal{F}$ is formalizing if it formalizes any f as above.*

4. *We say that $\mathcal{F}$ is v-formalizing if it v-formalizes any f as above.*

The previous technical definition will be used extensively, because it gives an abstract criterion to verify that a v-sheaf is v-locally formal.

Lemma 1.4.8. *The following statements hold:*

1. *The v-sheaf $\mathbb{Z}_p^\diamondsuit$ is formalizing.*

2. *$\mathrm{Spd}(B, B)$ is formalizing for any formal Huber pair over $\mathbb{Z}_p$.*

3. *A small v-sheaf $\mathcal{F}$ is v-formalizing if and only if it is v-locally formal.*

Proof. Given an affinoid perfectoid $\mathrm{Spa}(R, R^+)$ in characteristic p and an untilt $\iota : (R^\sharp)^\flat \to R$ we need to produce a natural transformation $\mathrm{Spd}(R^+, R^+) \to \mathbb{Z}_p^\diamondsuit$ for which the composition with the canonical map $\mathrm{Spa}(R, R^+) \to \mathrm{Spd}(R^+, R^+)$ gets mapped to $R^\sharp$. Let $\xi = p + [\varpi]\alpha$ be a generator of the kernel of $W(R^+) \to (R^\sharp)^+$, where $\varpi \in R^+$ denotes a pseudo-uniformizer and $\alpha \in W(R^+)$. Let $\mathrm{Spa}(A, A^+)$ be some other affinoid perfectoid space in characteristic p. Recall that, since R^+ is in characteristic p

$$\mathrm{Spd}(R^+, R^+)(A, A^+) = \{f : \mathrm{Spa}(A, A^+) \to \mathrm{Spa}(R^+, R^+)\}$$

Consider the following construction, take the map of topological rings $f^* : R^+ \to A^+$ defined by f, apply the Witt vector functor to f^* to get $W(f^*) : W(R^+) \to W(A^+)$ and consider the element $W(f^*)(\xi) \in W(A^+)$. We claim that $W(f^*)(\xi)$ is primitive of degree 1 (See [53] 6.2.8) and defines an untilt of $\mathrm{Spa}(A, A^+)$ over $\mathrm{Spa}(R^{\sharp^+}, R^{\sharp^+})$. Indeed $W(f^*)(\xi) = p + [f^*(\varpi)]f^*(\alpha)$ and it is enough to prove that there is a pseudo-uniformizer ϖ_A that divides $f^*(\varpi)$. This follows from the fact that $f^*(\varpi)$ is topologically nilpotent.

For the second claim, if we fix an untilt and a morphism $\mathrm{Spa}(R^\sharp, R^{\sharp^+}) \to \mathrm{Spa}(B, B)$ we can promote this to a morphism $\mathrm{Spa}(R^{\sharp,+}, R^{\sharp,+}) \to \mathrm{Spa}(B, B)$ and induce a formalization $\mathrm{Spd}(R^{\sharp,+}, R^{\sharp,+}) \to \mathrm{Spd}(B, B)$.

For the third claim, assume that $\mathcal{F}$ is v-formalizing. Since it is small there is a set I and a surjective map by a union of affinoid perfectoid spaces $\coprod_{i\in I} \mathrm{Spa}(R_i, {R_i}^+) \to \mathcal{F}$. After refining this cover we may assume that each of the maps $\mathrm{Spa}(R_i, {R_i}^+) \to \mathcal{F}$ formalizes to a map $\mathrm{Spd}(R_i^+, R_i^+) \to \mathcal{F}$, then $\coprod_{i\in I} \mathrm{Spd}(R_i^+, R_i^+) \to \mathcal{F}$ is also surjective proving that $\mathcal{F}$ is v-locally formal. If $\mathcal{F}$ is v-locally formal a map $\mathrm{Spa}(R, R^+) \to \mathcal{F}$ will v-locally factor through a map $\mathrm{Spa}(R, R^+) \to \mathrm{Spd}(B_i, B_i)$. By the second claim this map formalizes $\mathrm{Spd}(R^+, R^+) \to \mathrm{Spd}(B_i, B_i)$ and the composition to $\mathcal{F}$ is a formalization of the original map.

□

Since the notions of being v-locally formal and being v-formalizing are equivalent we will use them interchangeably, without mentioning it.

Proposition 1.4.9. *The following properties are easy to verify.*

1. *If $f : \mathcal{F} \to \mathcal{G}$ is a surjective map of small v-sheaves and $\mathcal{F}$ is v-formalizing then $\mathcal{G}$ is v-formalizing.*

2. *If $\mathrm{Spec}(R) \in \mathrm{PCAlg}_{\mathbb{F}_p}^{op}$ then $\mathrm{Spec}(R)^\diamondsuit$ is formalizing.*

3. *If $X \in \widetilde{\mathrm{SchPerf}}$ then $X^\diamondsuit$ is v-formalizing by lemma 1.3.32.*

4. *Non-empty v-formalizing v-sheaves have non-empty reduction. Consequently, diamonds are not v-formalizing.*

5. *If $\mathcal{F}$ formalizes $f : \mathrm{Spa}(A, A^+) \to \mathcal{F}$ then $\mathcal{F}$ formalizes $f \circ g$ for any map $g : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$*

Proposition 1.4.10. *Let $\mathcal{F}$ be a small v-sheaf, and $f : \mathrm{Spa}(R, R^+) \to \mathcal{F}$ a map with $\mathrm{Spa}(R, R^+)$ affinoid perfectoid in characteristic p. If $\mathcal{F}$ is formally separated then f admits at most one formalization.*

Proof. Suppose we are given two formalizations $g_i : \mathrm{Spd}(R^+, R^+) \to \mathcal{F}$ that agree on $\mathrm{Spa}(R, R^+)$. We get a map $(g_1, g_2) : \mathrm{Spd}(R^+, R^+) \to \mathcal{F} \times \mathcal{F}$, and we can pullback along the diagonal $\Delta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ to get $\mathcal{G} \subseteq \mathrm{Spd}(R^+, R^+)$ a formally closed subsheaf. We want to prove that $\mathcal{G} = \mathrm{Spd}(R^+, R^+)$, and to do so it is enough to prove the equality at the level of topological spaces, $|\mathcal{G}| = \mathrm{Spo}(R^+, R^+)$. Moreover, since $|\mathrm{Spa}(R, R^+)| \subseteq |\mathcal{G}|$ and $\mathrm{Spo}(R^+, R^+) = \mathrm{Spa}(R, R^+) \cup |\mathrm{Spec}(R^+_{\mathrm{red}})^\diamond|$ it is enough to prove $|(\mathcal{G}^{\mathrm{red}})^\diamond| = |\mathrm{Spec}(R^+_{\mathrm{red}})^\diamond|$. We warn the reader that we can't use a direct density argument because although $\mathrm{Spa}(R, R^+)$ is dense in $\mathrm{Spa}(R^+, R^+)$, it is no longer dense in $\mathrm{Spo}(R^+, R^+)$.

We first deal with the case in which C is a non-Archimedean field and $C^+ \subseteq C$ is an open and bounded valuation subring. Let $k^+ = C^+_{\mathrm{red}}$ and $k = Frac(k^+)$, we have that $\mathrm{Spec}(k^+) = \mathrm{Spd}(C^+, C^+)^{\mathrm{red}}$ and by lemma 1.3.33 $(\mathcal{G}^{\mathrm{red}})^\diamond = \mathrm{Spec}(k^+/I)^\diamond$ for some ideal I. On the other hand, since $\mathrm{Spa}(C, C^+) \subseteq \mathcal{G}$ and $|\mathcal{G}|$ is closed in $\mathrm{Spo}(C^+, C^+)$, $|\mathcal{G}|$ contains the formal specialization of $\mathrm{Spa}(C, O_C)$ in $\mathrm{Spo}(C^+, C^+)$, this corresponds to the image of $\mathrm{Spec}(k)^\diamond$. By formal adicness $|(\mathcal{G}^{\mathrm{red}})^\diamond| = |\mathcal{G}| \cap |\mathrm{Spec}(k^+)^\diamond|$ and we can conclude that $\mathrm{Spec}(k)^\diamond \subseteq (\mathcal{G}^{\mathrm{red}})^\diamond$. This proves that $I = \{0\}$ and that $(\mathcal{G}^{\mathrm{red}})^\diamond = \mathrm{Spec}(k^+)^\diamond$ as we needed to show.

For the general case, we get that for every map $\mathrm{Spa}(C, C^+) \to \mathrm{Spa}(R, R^+)$ the canonical formalization $\mathrm{Spd}(C^+, C^+) \to \mathrm{Spd}(R^+, R^+)$ factors through $\mathcal{G}$. In particular, after taking reduction, the map $\mathrm{Spec}(k^+) \to \mathrm{Spec}(R^+_{\mathrm{red}})$ factors through $\mathcal{G}^{\mathrm{red}}$. This says that $|\mathcal{G}^{\mathrm{red}}|$ contains every point of $|\mathrm{Spec}(R^+_{\mathrm{red}})|$ in the image of the specialization map. By lemma 1.3.33 $\mathcal{G}^{\mathrm{red}} \to \mathrm{Spec}(R^+_{\mathrm{red}})$ is a closed immersion and by proposition 1.4.2 the specialization map is surjective, these two imply that $\mathcal{G}^{\mathrm{red}} = \mathrm{Spec}(R^+_{\mathrm{red}})$. This also shows that $|(\mathcal{G}^{\mathrm{red}})^\diamond| = |\mathrm{Spec}(R^+_{\mathrm{red}})^\diamond|$ and concludes the proof. □

Proposition 1.4.11. *The following statements hold:*

1. *Given two maps of v-sheaves $\mathcal{F} \to \mathcal{H}$, $\mathcal{G} \to \mathcal{H}$ if $\mathcal{F}$ and $\mathcal{G}$ are v-formalizing and $\mathcal{H}$ is formally separated then $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ is v-formalizing.*

2. *The subcategory of v-sheaves that are v-formalizing and formally separated is stable under fiber product and contains $\mathbb{Z}_p^\diamond$.*

Proof. Given a map $\mathrm{Spa}(A, A^+) \to \mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ we can find a cover $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ for which the compositions with the projections to $\mathcal{F}$ and $\mathcal{G}$ are both formalizable. By formal separatedness any pair of choices of formalizations $\mathrm{Spd}(B^+, B^+) \to \mathcal{G}$ and to $\mathrm{Spd}(B^+, B^+) \to \mathcal{F}$ define the same formalization to $\mathcal{H}$ and a map to $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$. The second claim follows from the stability of separatedness by basechange and composition, from lemma 1.3.30 and from lemma 1.3.32. Indeed, we need to prove that $(\mathcal{F}^{\mathrm{red}})^\diamond \times_{(\mathcal{H}^{\mathrm{red}})^\diamond} (\mathcal{G}^{\mathrm{red}})^\diamond$ is a subsheaf of $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$, but this follows from knowing that $\mathcal{F}^{\mathrm{red}}$ (respectively $\mathcal{H}$, $\mathcal{G}$) is a subsheaf of $\mathcal{F}$ (respectively $\mathcal{H}$, $\mathcal{G}$). □

Definition 1.4.12. *Let $\mathcal{F}$ be a small v-sheaf, we say that it is specializing if it is formally separated and v-locally formal.*

Definition 1.4.13. *Let $\mathcal{F}$ be a specializing v-sheaf and let $f : \coprod_{i \in I} \mathrm{Spd}(B_i, B_i) \to \mathcal{F}$ be a surjective map. The specialization map for $\mathcal{F}$, denoted $\mathrm{sp}_{\mathcal{F}}$, is the unique map $\mathrm{sp}_{\mathcal{F}} : |\mathcal{F}| \to |\mathcal{F}^{\mathrm{red}}|$ making the following diagram commutative:*

$$\begin{tikzcd} \coprod_{i \in I} \mid \mathrm{Spd}(B_i, B_i) \mid \arrow[r, "f"] \arrow[d, "\mathrm{sp}_{B_i}"] & \mid \mathcal{F} \mid \arrow[d] \\ \coprod_{i \in I} \mid \mathrm{Spec}((B_i)_{\mathrm{red}}) \mid \arrow[r, "|f^{\mathrm{red}}|"] & \mid \mathcal{F}^{\mathrm{red}} \mid \end{tikzcd}$$

Remark 1.4.14. *We use proposition 1.4.10 to prove that this map of sets is well defined and does not depend on the choices taken. Indeed, given a point $[x] \in |\mathcal{F}|$ we may take a formalizable representative $x : \mathrm{Spa}(K_x, K_x{}^+) \to \mathcal{F}$. We take its unique formalization $\mathrm{Spd}(K_x^+, K_x^+) \to \mathcal{F}$ and we apply the reduction functor to this map. We obtain a map $\mathrm{Spec}((K_x^+)_{\mathrm{red}}) \to \mathcal{F}^{red}$, and the maximal ideal of $(K_x^+)_{\mathrm{red}}$ maps to a point in $|\mathcal{F}^{\mathrm{red}}|$, $\mathrm{sp}_{\mathcal{F}}([x])$ is this point. Suppose $y : \mathrm{Spa}(K_y, K_y{}^+) \to \mathcal{F}$ is another formalizable map with $[x] = [y]$ after replacing y by a cover we may assume that the map factors as $\mathrm{Spa}(K_y, K_y{}^+) \to \mathrm{Spa}(K_x, K_x{}^+) \to \mathcal{F}$. Since formalizations are unique we get a map*

$$\mathrm{Spd}(K_y^+, K_y^+) \to \mathrm{Spd}(K_x^+, K_x^+) \to \mathcal{F}$$

and the maximal ideal of $(K_y^+)_{\mathrm{red}}$ maps to the maximal ideal of $(K_x^+)_{\mathrm{red}}$

Proposition 1.4.15. *For any specializing v-sheaf $\mathcal{F}$ the specialization map $\mathrm{sp}_{\mathcal{F}} : |\mathcal{F}| \to |\mathcal{F}^{\mathrm{red}}|$ is continuous. Moreover, this construction is functorial in the category of specializing v-sheaves.*

Proof. We prove functoriality first, take a map of v-sheaves as above $g : \mathcal{F} \to \mathcal{G}$. Given a point $x : \mathrm{Spa}(K, K^+) \to \mathcal{F}$ the image in $|\mathcal{G}|$ is given by composition. A formalization for x gives a formalization for $g(x)$ and we get maps $\mathrm{Spec}(K^+_{\mathrm{red}}) \to \mathcal{F}^{\mathrm{red}} \to \mathcal{G}^{\mathrm{red}}$, we get that $g^{\mathrm{red}}(\mathrm{sp}_{\mathcal{F}}(x)) = \mathrm{sp}_{\mathcal{G}}(g(x))$.

Let us prove continuity, take a cover $f : \coprod_{i \in I} \mathrm{Spd}(B_i, B_i) \to \mathcal{F}$ with each (B_i, B_i) a formal Huber pair. By definition we get the following commutative diagram:

$$\begin{tikzcd} \mid \coprod_{i \in I} \mathrm{Spd}(B_i, B_i) \mid \arrow[r, "f"] \arrow[d, "\mathrm{sp}_{B_i}"] & \mid \mathcal{F} \mid \arrow[d, "\mathrm{sp}_{\mathcal{F}}"] \\ \mid \mathrm{Spec}((B_i)_{\mathrm{red}}) \mid \arrow[r, "f^{\mathrm{red}}"] & \mid \mathcal{F}^{\mathrm{red}} \mid \end{tikzcd}$$

The map f^{red} is continuous by proposition 1.3.15, the map f is continuous and a quotient map, and the maps sp_{B_i} are continuous by proposition 1.4.2. Since the diagram is commutative, the map $\mathrm{sp}_{\mathcal{F}}$ is also continuous. $\square$

1.4.3 Kimberlites, formal schemes and tubular neighborhoods

To prove pleasant properties of the specialization map we need to restrict our discussion to certain types of specializing v-sheaves.

Definition 1.4.16. *1. A pre-kimberlite is a small v-sheaf $\mathcal{F}$ such that:*

a) $\mathcal{F}$ is specializing.

b) $\mathcal{F}^{\mathrm{red}}$ is represented by a scheme.

c) The map $(\mathcal{F}^{\mathrm{red}})^{\diamond} \to \mathcal{F}$ coming from adjunction is a closed immersion.

2. For a pre-kimberlite $\mathcal{F}$, we define the analytic locus $\mathcal{F}^{an}$ as the open subsheaf $\mathcal{F} \setminus \mathcal{F}^{\mathrm{red}}$. If $\mathcal{F}^{an}$ is a locally spatial diamond we say that $\mathcal{F}$ is a kimberlite.

There are situations in which we will be interested in studying the specialization map when restricted to a proper subset of the analytic locus, for this reason we consider the following slightly more general concept.

Definition 1.4.17. *A smelted kimberlite is a pair $(\mathcal{F}, \mathscr{D})$ where $\mathcal{F}$ is a pre-kimberlite and $\mathscr{D} \subseteq \mathcal{F}^{an}$ is an open subsheaf such that $\mathscr{D}$ is a locally spatial diamond. A morphism of smelted kimberlites $f : (\mathcal{F}_1, \mathscr{D}_1) \to (\mathcal{F}_2, \mathscr{D}_2)$ is a morphism of v-sheaves $f : \mathcal{F}_1 \to \mathcal{F}_2$ such that $f(\mathscr{D}_1) \subseteq \mathscr{D}_2$.*

If we are given a kimberlite $\mathcal{F}$ over $\mathbb{Z}_p^{\diamond}$ there are two different smelted kimberlites that one can naturally associate to $\mathcal{F}$. That is, $(\mathcal{F}, \mathcal{F}^{an})$ and $(\mathcal{F}, \mathcal{F} \times_{\mathbb{Z}_p^{\diamond}} \mathbb{Q}_p^{\diamond})$. These two will only coincide if $\mathcal{F} \to \mathbb{Z}_p^{\diamond}$ is formally adic. We will use the following definition to abbreviate some sentences.

Definition 1.4.18. *A kimberlite (respectively prekimberlite) $\mathcal{F}$ together with a formally adic map $\mathcal{F} \to \mathbb{Z}_p^{\diamond}$ is said to be a p-adic kimberlite (respectively p-adic prekimberlite). Given a map of sheaves $\mathcal{F} \to \mathbb{Z}_p^{\diamond}$ we denote by $\mathcal{F}_\eta$ the basechange $\mathcal{F} \times_{\mathbb{Z}_p^{\diamond}} \mathbb{Q}_p^{\diamond}$. A p-smelted kimberlite is a prekimberlite $\mathcal{F}$ over $\mathbb{Z}_p^{\diamond}$ for which $(\mathcal{F}, \mathcal{F}_\eta)$ is a smelted kimberlite.*

Definition 1.4.19. *Given a smelted kimberlite $\mathcal{K} = (\mathcal{F}, \mathscr{D})$ we define the map of topological spaces $\mathrm{sp}_{\mathcal{K}} : |\mathscr{D}| \to |\mathcal{F}^{\mathrm{red}}|$ as the composition of $|\mathscr{D}| \to |\mathcal{F}| \xrightarrow{\mathrm{sp}_{\mathcal{F}}} |\mathcal{F}^{\mathrm{red}}|$. For a kimberlite $\mathcal{F}$ we abbreviate $\mathrm{sp}_{(\mathcal{F}, \mathcal{F}^{an})}$ by $\mathrm{sp}_{\mathcal{F}^{an}}$.*

Proposition 1.4.20. *Let $\mathcal{K} = (\mathcal{F}, \mathscr{D})$ be a smelted kimberlite, then $\mathrm{sp}_{\mathcal{K}}$ is a spectral map of locally spectral spaces. The construction of $\mathrm{sp}_{\mathcal{K}}$ is functorial in the category of smelted kimberlites.*

Proof. Continuity and functoriality follows directly from proposition 1.4.15. We need to prove that this map is also continuous for the constructible topology. Since it is enough to prove continuity on an open cover of $|\mathscr{D}|$, we may assume that $\mathscr{D}$ is a spatial diamond. We cover $\mathscr{D}$ by an affinoid perfectoid space $X = \mathrm{Spa}(A, A^+)$. Consider the diagram,

$$\begin{tikzcd} \mid \mathrm{Spa}(A,A^+) \mid^{cons} \arrow[r, "g"] \arrow[d, "\mathrm{sp}_X"] & \mid \mathscr{D} \mid^{cons} \arrow[d, "\mathrm{sp}_{\mathcal{K}}"] \\ \mid \mathrm{Spec}(A^+_{\mathrm{red}})^{\mathrm{red}} \mid^{cons} \arrow[r, "g^{\mathrm{red}}"] & \mid \mathcal{F}^{\mathrm{red}} \mid^{cons} \end{tikzcd}$$

where the spaces are given the constructible topology. Since $\mathcal{F}^{\mathrm{red}}$ is represented by a scheme proposition 1.3.17 implies that g^{red} is continuous for the constructible topology. Indeed, morphisms of schemes induce spectral maps. Similarly, the map sp_X is continuous and since X is a spatial diamond proposition 1.1.26 shows that the map g is also continuous. Moreover, g gives a surjective map of compact spaces and is consequently a quotient map. Since the diagram commutes, $\mathrm{sp}_{\mathcal{K}}$ is continuous for the patch topology. $\square$

The author thinks of kimberlites as a natural category in which one can consider "integral models" for diamonds. In what follows we will prove that v-sheaf associated to a separated formal scheme is a kimberlite.

For this one has to choose conventions carefully of what it means to be a "formal scheme". We take the convention given [52] section 2.2.

Convention 1. *Denote by* $\mathrm{Nilp}_{\mathbb{Z}_p}$ *the category of algebras in which* p *is nilpotent, and endow* $\mathrm{Nilp}_{\mathbb{Z}_p}^{op}$ *with the structure of a site by giving it the Zariski topology. By a formal scheme* $\mathfrak{X}$ *over* $\mathbb{Z}_p$ *we mean a Zariski sheaf on* $\mathrm{Nilp}_{\mathbb{Z}_p}^{op}$ *which is Zariski locally of the form* $\mathrm{Spf}(A)$*. Here* A *is a topological ring given the* I*-adic topology for a finitely generated ideal of* A *containing* p*, and* $\mathrm{Spf}(A)$ *denotes the functor* $\mathrm{Spec}(B) \mapsto \varinjlim_n Hom(A/I^n, B)$*.*

For a formal scheme $\mathfrak{X}$ *over* $\mathbb{Z}_p$ *we let* $\mathfrak{X}_{\mathrm{red}}$ *denote its reduction in the sense of formal schemes (See [56] Tag 0AIN). Recall that this is a sheaf in* $\mathrm{Nilp}_{\mathbb{Z}_p}^{op}$ *which is representable by a scheme. Moreover, the map* $\mathfrak{X}_{\mathrm{red}} \to \mathfrak{X}$ *is relatively representable in schemes, it is a closed immersion and for any open* $\mathrm{Spf}(A) \subseteq \mathfrak{X}$ *the pullback to* $\mathfrak{X}_{\mathrm{red}}$ *is given by the reduced subscheme of* $\mathrm{Spec}(A/I)$ *(for an ideal of definition* $I \subseteq A$*).*

We say that $\mathfrak{X}$ *is separated if* $\mathfrak{X}_{\mathrm{red}}$ *is a separated scheme (See [56] Tag 0AJ7).*

Recall the following result of Scholze and Weinstein

Proposition 1.4.21. *(See [52] 2.2.1) The functor* $\mathrm{Spf}(A) \mapsto \mathrm{Spa}(A, A)$ *extends to a fully faithful functor* $\mathfrak{X} \mapsto \mathfrak{X}^{ad}$ *from formal schemes over* $\mathbb{Z}_p$ *as in convention 1 to the category of pre-adic spaces.*

Remark 1.4.22. *We warn that what is called adic spaces in [52] is what we call pre-adic spaces here and in [53].*

Proposition 1.4.23. *If* $\mathfrak{X}$ *is a separated formal scheme over* $\mathbb{Z}_p$ *as in convention 1 then* $(\mathfrak{X}^{ad})^\lozenge$ *is a kimberlite.*

Proof. Let $X = \mathfrak{X}^{ad}$ and let $W = X^{na}$, we have that $W = (\mathfrak{X}_{red})^{ad}$. Clearly $X^\lozenge$ is v-locally formal since it is open locally of the form $\mathrm{Spd}(B, B)$. By proposition 1.3.20 we have $(W^\lozenge)^{\mathrm{red}} = (X^\lozenge)^{\mathrm{red}}$ which is the perfection of $\mathfrak{X}_{red}$. The adjunction morphism agrees with the map $W^\diamond \to X^\diamond$ which by proposition 1.1.31 is a closed immersion. Moreover, this says $X^\diamond \setminus ((X^\diamond)^{\mathrm{red}})^\diamond = (X^{an})^\diamond$ so by proposition 1.1.31 this is a locally spatial diamond.

The only thing left to prove is that $X^\diamond \to \mathbb{Z}_p^\diamond$ is separated, we fist prove that $X^\diamond$ is quasi-separated. Let $Z = \mathrm{Spa}(R, R^+)$ be a strictly totally disconnected space and take a map $f : Z \to X^\diamond \times_{\mathbb{Z}_p^\diamond} X^\diamond$. Since Z splits any open cover we may assume that f factors through an open neighborhood of the form $\mathrm{Spd}(B_1, B_1) \times_{\mathbb{Z}_p^\diamond} \mathrm{Spd}(B_2, B_2)$ for an open subset $\mathrm{Spf}(B_1) \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathrm{Spf}(B_2) \subseteq \mathfrak{X} \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathfrak{X}$. Consider the following basechange diagrams, where $Y = \mathfrak{Y}^{ad}$

$$\begin{tikzcd}
\mathfrak{Y} \arrow[r] \arrow[d] & \mathrm{Spf}(B_1) \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathrm{Spf}(B_2) \arrow[d] \\
\mathfrak{X} \arrow[r] & \mathfrak{X} \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathfrak{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r] \arrow[d] & \mathrm{Spa}(B_1, B_1) \times_{\mathbb{Z}_p} \mathrm{Spa}(B_2, B_2) \arrow[d] \\
X \arrow[r] & X \times_{\mathbb{Z}_p} X
\end{tikzcd}$$

Since $\mathfrak{X}$ is separated $\mathfrak{Y}$ is quasi-compact. This implies that Y admits a finite open cover of the form $\coprod_{i=1}^n \mathrm{Spa}(A_i, A_i) \to Y$. Moreover, the diagonal map $X \to X \times_{\mathbb{Z}_p} X$ is adic (sends analytic points to analytic points). Indeed, composing the diagonal map with one of the projections gives the identity. Since open immersions preserve adicness the maps of pre-adic spaces $\mathrm{Spa}(A_i, A_i) \to \mathrm{Spa}(B_1, B_1) \times_{\mathbb{Z}_p} \mathrm{Spa}(B_2, B_2)$ are adic. By lemma 1.2.26 the maps $\mathrm{Spd}(A_i, A_i) \to \mathrm{Spd}(B_1, B_1) \times_{\mathbb{Z}_p} \mathrm{Spd}(B_2, B_2)$ are quasi-compact, which proves that $Y^\diamond \to \mathrm{Spd}(B_1, B_1) \times_{\mathbb{Z}_p} \mathrm{Spd}(B_2, B_2)$ and any basechange of it is also quasi-compact. After proving that the map $X^\diamond \to \mathbb{Z}_p^\diamond$ is quasi-separated we may use the valuative criterion of separatedness (See [51] 10.9). We must prove that for a perfectoid field and a map $\mathrm{Spa}(K, O_K) \to X^\diamond$ there is at most one extension to $\mathrm{Spa}(K, K^+) \to X^\diamond$ where $K^+ \subseteq O_K$ is an open and bounded valuation subring. Maps $\mathrm{Spa}(K, K^+) \to X^\diamond$ are in bijection with maps $\mathrm{Spf}(K^+) \to \mathfrak{X}$. On the other hand, maps $g : \mathrm{Spf}(K^+) \to \mathfrak{X}$ are in bijection with pairs (g_η, g_s) where $g_\eta : \mathrm{Spf}(O_K) \to \mathfrak{X}$, $g_s : \mathrm{Spec}(K^+/K^{\circ\circ}) \to \mathfrak{X}_{red}$ and such that $g_\eta = g_s$ when we restrict the maps to to $\mathrm{Spec}(O_K/K^{\circ\circ})$. At this point we may use the valuative criterion of separatedness of $\mathfrak{X}_{red}$.

□

The following concept is central to our purposes.

Definition 1.4.24. *Let $\mathcal{F}$ be a prekimberlite and let $S \subseteq \mathcal{F}^{\mathrm{red}}$ be a locally closed immersion of schemes.*

1. *We define the tubular neighborhood of S on $\mathcal{F}$, denoted $\widehat{\mathcal{F}}_{/S}$, as the sub-v-sheaf of $\mathcal{F}$ defined by the following Cartesian diagram:*

$$\begin{tikzcd}
\widehat{\mathcal{F}}_{/S} \arrow[r] \arrow[dd] & \mathcal{F} \arrow[d] \\
 & \underline{\mid \mathcal{F} \mid} \arrow[d, "\mathrm{sp}_{\mathcal{F}}"] \\
\underline{\mid S \mid} \arrow[r] & \underline{\mid \mathcal{F}^{\mathrm{red}} \mid}
\end{tikzcd}$$

2. *If $\mathcal{K} = (\mathcal{F}, \mathscr{D})$ is a smelted kimberlite we let $\widehat{\mathscr{D}}_{/S}$ be $\widehat{\mathcal{F}}_{/S} \cap \mathscr{D}$ and we refer to this sheaf as the smelted tubular neighborhood.*

3. *If $\mathcal{F}$ comes equipped with a map to $\mathbb{Z}_p^\diamondsuit$ (not necessarily formally adic) we define the p-adic tubular neighborhood of $\mathcal{F}$, denoted $(\widehat{\mathcal{F}}_{/S})_\eta$, as the basechange $\widehat{\mathcal{F}}_{/S} \times_{\mathbb{Z}_p^\diamondsuit} \mathbb{Q}_p^\diamondsuit$.*

Intuitively speaking, $\widehat{\mathcal{F}}_{/S}$ is the subsheaf of points whose specialization map factors through S. This notion generalizes completions along a closed subscheme in formal geometry:

Proposition 1.4.25. *Suppose (A, A) is a formal Huber pair over $\mathbb{Z}_p$ with ideal of definition I. Let $J \subseteq A$ be a finitely generated ideal containing I and B the completion of A with respect to J. The closed immersion of schemes $S = \mathrm{Spec}(B_{\mathrm{red}}) \to \mathrm{Spec}(A_{\mathrm{red}})$, induces an identification $\widehat{\mathrm{Spd}(A, A)}_{/S} = \mathrm{Spd}(B, B)$.*

Proof. Let $S = \mathrm{Spec}(B_{\mathrm{red}})$ and $T = \mathrm{Spec}(A_{\mathrm{red}})$. The reduction of the map $\mathrm{Spd}(B, B) \to \mathrm{Spd}(A, A)$ induces the map $S \to T$. Since specialization is functorial any point coming from $\mathrm{Spd}(B, B)$ has to specialize to S. Consequently the map factors as $\mathrm{Spd}(B, B) \to \widehat{\mathrm{Spd}(A, A)}_{/S} \to \mathrm{Spd}(A, A)$. Since A is dense in B, it is easy to see that this map is an injection. To prove surjectivity onto $\widehat{\mathrm{Spd}(A, A)}_{/S}$, suppose we have a map $f : A \to R^+$ for which the induced map $f : \mathrm{Spec}(R^+_{\mathrm{red}}) \to \mathrm{Spec}(A_{\mathrm{red}})$ factors through $|S|$. Then for every $a \in J$ the element $f(a)$ is nilpotent in $\mathrm{Spec}(R^+/\varpi^n)$. Since J is finitely generated there is an m for which $J^m \subseteq (\varpi^n)$ in R^+. This proves that the map $f : A \to R^+$ is continuous for the J-adic topology on A. Since R^+ is complete the map $f : A \to R^+$ factors through B, which proves that any map $\mathrm{Spa}(R, R^+) \to \widehat{\mathrm{Spd}(A, A)}_{/S}$ factors through a map to $\mathrm{Spd}(B, B)$. $\square$

Proposition 1.4.26. *Let $f : \mathcal{G} \to \mathcal{F}$ be a morphism prekimberlites and let $S \subseteq |\mathcal{F}^{\mathrm{red}}|$ a locally closed subscheme. If we define $T = S \times_{\mathcal{F}^{\mathrm{red}}} \mathcal{G}^{\mathrm{red}}$, then $\widehat{\mathcal{F}}_{/S} \times_{\mathcal{F}} \mathcal{G} = \widehat{\mathcal{G}}_{/T}$. In particular, a map of prekimberlites $\mathcal{G} \to \mathcal{F}$ factors through $\widehat{\mathcal{F}}_{/S}$ if and only if $\mathcal{G}^{\mathrm{red}} \to \mathcal{F}^{\mathrm{red}}$ factors through S.*

Proof. Since S is a locally closed immersion we have $|T| = |S| \times_{|\mathcal{F}^{\mathrm{red}}|} |\mathcal{G}^{\mathrm{red}}|$. We can look at the following commutative diagram:

$$\begin{tikzcd}
\mathcal{G} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\
\underline{\mid G^{\mathrm{red}} \mid} \arrow[r] & \underline{\mid \mathcal{F}^{\mathrm{red}} \mid}
\end{tikzcd}$$

The first claim follows by basechanging this diagram by the map $\underline{|S|} \to \underline{|\mathcal{F}^{\mathrm{red}}|}$. For the second claim, observe that the map $\mathcal{G} \to \mathcal{F}$ factoring through $\widehat{\mathcal{F}}_{/S}$ is equivalent to $\mathcal{G} \times_{\mathcal{F}} \widehat{\mathcal{F}}_{/S} = \mathcal{G}$. By the first claim this is equivalent to $\widehat{\mathcal{G}}_{/T} = \mathcal{G}$, which happens if and only if $T = \mathcal{G}^{\mathrm{red}}$ and $\mathcal{G}^{\mathrm{red}} \to \mathcal{F}^{\mathrm{red}}$ factors through S. $\square$

Remark 1.4.27. *Let $\mathcal{F}$ be a prekimberlite and $S \subseteq \mathcal{F}^{\mathrm{red}}$ locally closed subset. One can prove that the v-sheaf $\widehat{\mathcal{F}}_{/S}$ is a small v-sheaf but this is not automatic. The problem is that the v-sheaf $\underline{T}$ is not small whenever the topological space T does not satisfy the separation axiom $T1$.*

Proposition 1.4.28. *Let $\mathcal{F}$ be a prekimberlite and let $S \subseteq |\mathcal{F}^{\mathrm{red}}|$ a locally closed subset, then $\widehat{\mathcal{F}}_{/S}$ is a prekimberlite and $(\widehat{\mathcal{F}}_{/S})^{\mathrm{red}} = S$.*

Proof. The formula $(\widehat{\mathcal{F}}_{/S})^{\mathrm{red}} = S$ follows easily from observing that by proposition 1.4.26 a map $\mathrm{Spec}(A)^{\diamond} \to \mathcal{F}$ factors through $\widehat{\mathcal{F}}_{/S}$ if and only if the map obtained by adjunction $\mathrm{Spec}(A) \to \mathcal{F}^{\mathrm{red}}$ factors through S. Indeed, S and $\widehat{\mathcal{F}}^{\mathrm{red}}_{/S}$ represent the same functor in this case.

For the first claim, since $\widehat{\mathcal{F}}_{/S}$ is a subsheaf of a formally separated v-sheaf it is formally separated as well. To prove it is v-formalizing take a map $\mathrm{Spa}(R, R^+) \to \widehat{\mathcal{F}}_{/S} \subseteq \mathcal{F}$. After replacing $\mathrm{Spa}(R, R^+)$ by a v-cover if necessary we get a formalization $\mathrm{Spd}(R^+, R^+) \to \mathcal{F}$. By proposition 1.4.26 this formalization factors through $\widehat{\mathcal{F}}_{/S}$ if and only if $\mathrm{Spec}(R^+_{\mathrm{red}}) \to \mathcal{F}^{\mathrm{red}}$ factors through S. But this later condition holds since $\mathrm{Spa}(R, R^+) \to \mathcal{F}$ factors through $\widehat{\mathcal{F}}_{/S}$. To finish the proof we need to show that $S^{\diamond} \to \widehat{\mathcal{F}}_{/S}$ is a closed immersion. Consider the base change $\widehat{\mathcal{F}}_{/S} \times_{\mathcal{F}} (\mathcal{F}^{\mathrm{red}})^{\diamond}$. On one hand the projection to $\widehat{\mathcal{F}}_{/S}$ is a closed immersion, and on the other hand by proposition 1.4.26 this identifies with $(\widehat{(\mathcal{F}^{\mathrm{red}})^{\diamond}})_{/S}$. In case S is a closed subscheme of $\mathcal{F}^{\mathrm{red}}$ we have that the map of v-sheaves $S^{\diamond} \to (\mathcal{F}^{\mathrm{red}})^{\diamond}$ is proper so that $S^{\diamond} \to (\widehat{(\mathcal{F}^{\mathrm{red}})^{\diamond}})_{/S}$ is also a closed immersion. In case S is an open subscheme of $\mathcal{F}^{\mathrm{red}}$, we can verify $(\widehat{(\mathcal{F}^{\mathrm{red}})^{\diamond}})_{/S} = S^{\diamond}$. The general case follows from these two cases. $\square$

Whenever S is a constructible subset we can say more:

Proposition 1.4.29. *Let $\mathcal{F}$ be a prekimberlite, $S \subseteq |\mathcal{F}^{\mathrm{red}}|$ a locally closed constructible subset then:*

1. *The map $\widehat{\mathcal{F}}_{/S} \to \mathcal{F}$ is an open immersion.*

2. *If $\mathcal{K} = (\mathcal{F}, \mathscr{D})$ is a smelted kimberlite, then $\widehat{\mathcal{F}}_{/S} \cap \mathscr{D}$ is the open subsheaf corresponding to the interior of ${\mathrm{sp}_{\mathcal{K}}}^{-1}(S)$ in $|\mathscr{D}|$.*

Proof. For the first claim we begin by observing that the question is Zariski local in $\mathcal{F}^{\mathrm{red}}$. Indeed, an open cover $\coprod_{i\in I} U_i \to \mathcal{F}^{\mathrm{red}}$ induces an open cover $\coprod_{i\in I} \widehat{\mathcal{F}}_{/U_i} \to \mathcal{F}$. After localizing, we may assume that $\mathcal{F}^{\mathrm{red}} = \mathrm{Spec}(A)$ and that S is closed subset of $\mathrm{Spec}(A)$ that is open for the constructible topology. Write $S = \mathrm{Spec}(A/I)$ for $I \subseteq A$ an ideal, since we are only interested in S as a topological space, a compactness argument allows us to assume that I is finitely generated. Let $(i_1, \dots, i_n)$ be a list of generators for I, let (R, R^+) be a perfectoid Huber pair and $\mathrm{Spd}(R^+, R^+) \to \mathcal{F}$ a map. We can describe the basechange $X := \mathrm{Spd}(R^+, R^+) \times_{\mathcal{F}} \widehat{\mathcal{F}}_{/S}$ as follows. Let $\varpi \in R^+$ be a pseudo-uniformizer and $(j_1, \dots, j_n)$ a list of lifts of $(i_1, \dots i_n)$ in R^+_{red}. We claim X is the open subsheaf of $\mathrm{Spd}(R^+, R^+)$ defined by $\bigcap_{k=1}^n N_{j_k<<1}$. Indeed, by proposition 1.4.26 X is given by $\mathrm{Spd}\widehat{(R^+, R^+)}_{/V(I)}$ and by proposition 1.4.25 if we let B^+ be the completion of R^+ by the (I, ϖ)-adic topology then $X = \mathrm{Spd}(B^+, B^+)$. That $\mathrm{Spd}(B^+, B^+) = \bigcap_{k=1}^n N_{j_k<<1}$ is a direct consequence of lemma 1.2.23. Since $\mathcal{F}$ is v-formalizing every map $\mathrm{Spa}(R, R^+) \to \mathcal{F}$ factors through $\mathrm{Spd}(R^+, R^+)$ after replacing $\mathrm{Spa}(R, R^+)$ by a v-cover. In particular, the basechanges $\mathrm{Spa}(R, R^+) \times_{\mathcal{F}} \widehat{\mathcal{F}}_{/S}$ are open after taking a v-cover. By [51] 10.11 $\widehat{\mathcal{F}}_{/S} \to \mathcal{F}$ is open.

For the second claim let $T \subseteq \mathrm{sp}_{\mathcal{K}}{}^{-1}(S)$ be the largest subset stable under generization. We prove that $T \subseteq \widehat{\mathcal{F}}_{/S} \cap \mathscr{D}$ since we already have a chain of inclusions:

$$\widehat{\mathcal{F}}_{/S} \cap \mathscr{D} \subseteq (\mathrm{sp}_{\mathcal{K}}{}^{-1}(S))^{int} \subseteq T.$$

Take $x \in T$ and a formalizable geometric point $\iota_x : \mathrm{Spa}(C_x, C_x{}^+) \to \mathcal{F}$ over x. Since every generization of x is in $\mathrm{sp}_{\mathcal{K}}{}^{-1}(S)$ the map $\mathrm{Spec}((C_x^+)_{\mathrm{red}}) \to \mathcal{F}^{\mathrm{red}}$ factors through S so that the composition $|\mathrm{Spa}(C_x, C_x{}^+)| \to |\mathscr{D}| \to |\mathcal{F}^{\mathrm{red}}|$ factors through $|S|$, giving that ι_x factors through $\widehat{\mathcal{F}}_{/S}$.

□

Proposition 1.4.30. *Let $f : \mathcal{G} \to \mathcal{F}$ be a formally closed immersion of small v-sheaves. The following hold:*

1. *If $\mathcal{F}$ is a specializing v-sheaf, then $\mathcal{G}$ is a specializing v-sheaf.*
2. *If $\mathcal{F}$ is a prekimberlite, then $\mathcal{G}$ is a prekimberlite.*
3. *If $\mathcal{F}$ is a kimberlite, then $\mathcal{G}$ is a kimberlite.*
4. *If $(\mathcal{F}, \mathscr{D})$ forms a smelted kimberlite then $(\mathcal{G}, \mathcal{G} \cap \mathscr{D})$ forms a smelted kimberlite.*

Proof. Suppose $\mathcal{F}$ is specializing, since $\mathcal{G}$ is a subsheaf of $\mathcal{F}$ it is formally separated. Observe that for a perfectoid Huber pair (R, R^+) and a map $\mathrm{Spd}(R^+, R^+) \to \mathcal{F}$ the basechange $X := \mathcal{G} \times_{\mathcal{F}} \mathrm{Spd}(R^+, R^+)$ is a formally closed subsheaf of $\mathrm{Spd}(R^+, R^+)$. We may reason as in the proof of proposition 1.4.10 to conclude $X = \mathrm{Spd}(R^+, R^+)$ whenever $\mathrm{Spa}(R, R^+) \to \mathcal{F}$ factors through $\mathcal{G}$. This proves that $\mathcal{G}$ is also v-formalizing and a specializing sheaf. Suppose $\mathcal{F}$ is a prekimberlite, we have a commutative diagram:

$$\begin{tikzcd}
(\mathcal{G}^{\mathrm{red}})^{\diamond} \arrow[r] \arrow[d] & (\mathcal{F}^{\mathrm{red}})^{\diamond} \arrow[d] \\
\mathcal{G} \arrow[r] & \mathcal{F}
\end{tikzcd}$$

By formal adicness this diagram is Cartesian which gives that $(\mathcal{G}^{\mathrm{red}})^{\diamond} \to \mathcal{G}$ is a closed immersion. Since the map $(\mathcal{G}^{\mathrm{red}})^{\diamond} \to (\mathcal{F}^{\mathrm{red}})^{\diamond}$ is also a formally closed immersion by lemma 1.3.33 $\mathcal{G}^{\mathrm{red}}$ is represented by a closed subscheme of $\mathcal{F}$, finishing the proof that $\mathcal{G}$ is a prekimberlite. Suppose now that $\mathcal{F}$ is a kimberlite, then $\mathcal{G}^{an} = \mathcal{F}^{an} \times_{\mathcal{F}} \mathcal{G}$ and by ([51] 11.20) it is a locally spatial diamond, so $\mathcal{G}$ is a kimberlite. The same applies for $\mathcal{G} \times_{\mathcal{F}} \mathscr{D}$ in the smelted kimberlite case. $\square$

1.4.4 cJ-diamonds and rich kimberlites

Suppose we have a formal scheme $\mathcal{X}$ topologically of finite type over $\mathbb{Z}_p$, suppose we let X_η denote the generic fiber of $\mathcal{X}$ considered as an adic space over $\mathbb{Q}_p$ and suppose we let X^{red} denote the reduced special fiber of $\mathcal{X}$ considered as a scheme over $\mathbb{F}_p$. In this classical situation we have a specialization map $\mathrm{sp}_{X_\eta} : |X_\eta| \to |X^{\mathrm{red}}|$, and for a fixed closed point $x \in |X^{\mathrm{red}}|$ we have the following chain of inclusions $|(\widehat{\mathcal{X}}_{/x})_\eta| \subseteq {\mathrm{sp}_{X_\eta}}^{-1}(x) \subseteq |X_\eta|$. These inclusions satisfy that:

1. ${\mathrm{sp}_{X_\eta}}^{-1}(x)$ is a closed subset.
2. $|(\widehat{\mathcal{X}}_{/x})_\eta|$ is the interior of ${\mathrm{sp}_{X_\eta}}^{-1}(x)$ in $|X_\eta|$.
3. $|(\widehat{\mathcal{X}}_{/x})_\eta|$ is dense in ${\mathrm{sp}_{X_\eta}}^{-1}(x)$

The first two conditions generalize, by proposition 1.4.29, to the case of kimberlites for which closed points are constructible. In this section we give sufficient conditions that make a kimberlite have the third property as well. Before discussing these condition we give an example showing that some sort of finiteness hypothesis need to be imposed for the third property to hold.

Example 1.4.31. *Let C be a p-adic non-Archimedean field and C^+ an open and bounded valuation subring whose rank is strictly larger than 1. We have that sp_C is a homeomorphism between $\mathrm{Spa}(C, C^+)$ and $\mathrm{Spec}(C^+/C^{\circ\circ})$. In particular, if x denotes the closed point of $\mathrm{Spec}(C^+/C^{\circ\circ})$ then ${\mathrm{sp}_C}^{-1}(x)$ is the closed point of $y \in \mathrm{Spa}(C, C^+)$. The interior of $\{y\}$ is empty, therefore it is not a dense subset of $\{y\}$.*

Definition 1.4.32. *We say that a locally spatial diamond X is constructibly Jacobson if the subset of rank 1 points are dense for the constructible topology of $|X|$. Locally spatial diamonds with this property will be called cJ-diamonds.*

Proposition 1.4.33. *Suppose that $\mathcal{K} = (\mathcal{F}, \mathscr{D})$ is a smelted kimberlite with $\mathscr{D}$ a cJ-diamond, let $S \subseteq |\mathcal{F}|$ a constructible subset. Then $|\mathscr{D} \cap \widehat{\mathcal{F}}_{/S}|$ is dense in ${\mathrm{sp}_{\mathcal{K}}}^{-1}(S)$.*

Proof. By the proof of proposition 1.4.29, we know that $|\mathscr{D} \cap \widehat{\mathcal{F}}_{/S}|$ is the largest subset of ${\mathrm{sp}_{\mathcal{K}}}^{-1}(S)$ stable under generization. Since S is constructible and $\mathrm{sp}_{\mathcal{K}}$ is a spectral map, the set ${\mathrm{sp}_{\mathcal{K}}}^{-1}(S)$ is open in the constructible topology of $|\mathscr{D}|$ and rank 1 points contained in this set are dense in it. Since rank 1 points are stable under generization, they belong to $|\mathscr{D} \cap \widehat{\mathcal{F}}_{/S}|$. This proves that $|\mathscr{D} \cap \widehat{\mathcal{F}}_{/S}|$ is dense in ${\mathrm{sp}_{\mathcal{K}}}^{-1}(S)$ for the constructible topology, but the usual topology is coarser so it is dense for the usual topology as well. $\square$

We discuss some properties of this concept.

Proposition 1.4.34. *Let $f : X \to Y$ be a morphism of locally spatial diamonds the following hold:*

1. *Suppose that $|f|$ is a surjective map of topological spaces and that X is a cJ-diamond, then Y is a cJ-diamond.*

2. *Suppose that f is an open immersion and that Y is a cJ-diamond, then X is a cJ-diamond.*

3. *Suppose that f realizes X as a quasi-pro-étale $\underline{J}$-torsor over Y for some profinite group J and that X is a cJ-diamond, then Y is a cJ-diamond.*

4. *Suppose that f is étale and that Y is a cJ-diamond, then X is a cJ-diamond.*

Proof. Maps of locally spatial diamonds induce continuous spectral maps of locally spectral spaces. Surjective maps send dense subsets to dense subsets. Moreover, maps of locally spatial diamonds are generalizing which implies that rank 1 points can only map to rank 1 points. This proves the first claim.

Suppose now that Y is a cJ-diamond. If f is an open immersion, any open in the patch topology of X is also open in the patch topology of Y and contains a rank 1 point, this proves the second claim. Moreover this allow us to localize in the analytic topology, so we can assume for the rest of the argument X and Y are spatial.

If f is étale, by ([51] 11.31) locally for the analytic topology we can write f as the composition of an open immersion and a finite étale map. The category of finite étale morphisms over a fixed spatial diamond is a Galois category and using the first claim we may reduce to the case in which f is Galois with finite Galois group G. In this way, the fourth claim follows from the third.

In the setup of the third claim, we claim (and prove below) that f is an open mapping for the patch topology, this would finish the proof. Indeed, if a point y maps to x under a quasi-pro-étale map and x is rank 1 then y is also rank 1.

Let $J = \varprojlim_i J_i$ with J_i a cofiltered family of finite groups and denote by $f_i : X_i \to Y$ the induced J_i-torsors. We get action maps $J_i \times |X_i| \to |X_i|$ that are continuous for the discrete topology on J_i and the constructible topology on $|X_i|$. Moreover, for any set $S \subseteq |X_i|$ we have that $f_i^{-1}(f_i(S)) = J_i \cdot S$. Now, the formation of the patch topology on a spectral space commutes with limits along spectral maps. This gives an action map $J \times |X| \to |X|$ that is continuous when $|X|$ is given the patch topology and J is given its profinite topology. Let

$U \subseteq X$ be open in the constructible topology, then $f^{-1}(f(U)) = J \cdot U$ which is also open. The map $|f|^{cons} : |X|^{cons} \to |Y|^{cons}$ is a surjective continuous map of compact spaces, so it is a quotient map. Since $J \cdot U$ is open and saturated $f(J \cdot U) = f(U)$ is open as we wanted to show. $\square$

Let's recall the following theorem of Huber:

Theorem 1.4.35. *(See [23] Theorem 4.1) Let k be a complete field with respect to a rank 1 valuation, and let A be a k-algebra of topologically finite type over k. Then the subset $Max(A) \subseteq \mathrm{Spa}(A, A^\circ)$ is dense for the constructible topology.*

Huber's statement says something a bit stronger, but this weaker form of the statement is easier to state and the one we will use in applications.

Corollary 1.4.36. *If X is an adic space topologically of finite type over $\mathrm{Spa}(k, k^\circ)$, where $\mathrm{Spa}(k, k^\circ)$ is a non-Archimedean field over $\mathbb{Z}_p$. Then $X^\diamondsuit$ is a cJ-diamond.*

Proof. The claim is local on X so we can assume $X = \mathrm{Spa}(A, A^\circ)$ for a Tate algebra, A/k. In this case every point $Max(A)$, when considered as a valuation, is a rank 1 valuation. $\square$

Example 1.4.37. *The perfectoid unit ball $\mathbb{B}_n = \mathrm{Spa}(C\langle T_1^{\frac{1}{p^\infty}} \dots T_n^{\frac{1}{p^\infty}}\rangle, O_C\langle T_1^{\frac{1}{p^\infty}} \dots T_n^{\frac{1}{p^\infty}}\rangle)$ over a perfectoid field C of characteristic p, is a cJ-diamond. Indeed, we have the equality of diamonds*

$$\mathrm{Spa}(C\langle T_1 \cdots T_n\rangle, O_C\langle T_1 \cdots T_n\rangle)^\diamondsuit = \mathbb{B}_n,$$

and we may conclude by theorem 1.4.35.

Definition 1.4.38. *Let C be a perfectoid field in characteristic p and X a locally spatial diamond over $Spa(C, O_C)$. We say that X has "enough facets" over C if it admits a v-cover of the form $\coprod_{i \in I} \mathrm{Spd}(A_i, A_i^\circ) \to X$ where each A_i is an algebra topologically of finite type over C.*

Proposition 1.4.39. *Let X and Y be two locally spatial diamonds with enough facets over C, and let $C^\sharp$ denote an untilt of C. The following hold:*

1. *For any morphism of perfectoid fields $\mathrm{Spa}(C', O_{C'}) \to \mathrm{Spa}(C, O_C)$ the base change $X \times_{\mathrm{Spa}(C, O_C)} \mathrm{Spa}(C', O_{C'})$ has enough facets over C'.*
2. *The fiber product $X \times_{\mathrm{Spa}(C, O_C)} Y$ has enough facets over C.*
3. *X is a cJ-diamond.*
4. *If $X = \mathrm{Spd}(A, A^\circ)$ for a smooth and topologically of finite type $C^\sharp$-algebra A, then X has enough facets.*

Proof. Since the property of being topologically of finite type is stable under products and change of the base ground field one can prove easily the first two claims. The third claim follows from corollary 1.4.36 and proposition 1.4.34.

For the last claim, let $\mathbb{T}^n_{C^\sharp}$ denote $\mathrm{Spa}(C^\sharp\langle T_1^\pm, \dots T_n^\pm\rangle, O_{C^\sharp}\langle T_1^\pm, \dots T_n^\pm\rangle)$, and let $\widetilde{\mathbb{T}}^n_{C^\sharp} = \varprojlim_{T_i \mapsto T_i^p} \mathbb{T}^n_{C^\sharp}$ analogously for $\mathbb{T}^n_C$ and $\widetilde{\mathbb{T}}^n_C$. For any point $x \in \mathrm{Spa}(A, A^\circ)$ we may find an open neighborhood U of x together with an étale map $\eta : U \to \mathbb{T}^n_{C^\sharp}$. Let $\widetilde{U}$ denote the pullback of η along $\widetilde{\mathbb{T}}^n_{C^\sharp} \to \mathbb{T}^n_{C^\sharp}$, we get an étale map $\widetilde{U}^\flat \to \widetilde{\mathbb{T}}^n_C$. By the invariance of the étale site under perfection (see [51] lemma 15.6) $\widetilde{U}^\flat = U'^\diamondsuit$ for an adic space U' that is étale over $\mathbb{T}^n_C$. Now, U' admits an open cover of the form $\coprod_{i \in I} \mathrm{Spa}(A_i, A_i^\circ) \to U'$ with each A_i topologically of finite type over C. This gives a cover,

$$\coprod_{i \in I} \mathrm{Spd}(A_i, A_i^\circ) \to \widetilde{U}^\flat \to U^\diamondsuit.$$

□

We now define rich kimberlites, which are some of the kimberlites that will satisfy the third condition we discussed above.

Definition 1.4.40. *Let $\mathcal{F}$ be a prekimberlite and $\mathcal{K} = (\mathcal{F}, \mathscr{D})$ a smelted kimberlite.*

1. *We say that $\mathcal{K}$ is rich if the following conditions hold:*
   - *$\mathscr{D}$ is a cJ-diamond.*
   - *$|\mathcal{F}^{\mathrm{red}}|$ is a locally Noetherian topological space.*
   - *The specialization map $\mathrm{sp}_\mathcal{K} : |\mathscr{D}| \to |\mathcal{F}^{\mathrm{red}}|$ is specializing and a quotient map.*
2. *If $\mathcal{F}$ is a kimberlite we say it is rich if $(\mathcal{F}, \mathcal{F}^{an})$ is rich. If $\mathcal{F}$ is a p-smelted kimberlite we say it is rich if $(\mathcal{F}, \mathcal{F}_\eta)$ is rich.*

Remark 1.4.41. *To the author's knowledge, the theory of diamonds and v-sheaves doesn't have a good notion of what it means to be of "finite type". Being rich, is an ad hoc condition that is good enough for the applications that we have in mind.*

The following fact about rich smelted kimberlites is a crucial property that we use later on in our applications.

Proposition 1.4.42. *Let $\mathcal{K} = (\mathcal{F}, \mathscr{D})$ be a rich smelted kimberlite and suppose that for any closed point $x \in |\mathcal{F}^{\mathrm{red}}|$ the smelted tubular neighborhood $\widehat{\mathscr{D}}_{/x}$ is connected, then $\pi_0(\mathrm{sp}_\mathscr{D}) : \pi_0(|\mathscr{D}|) \to \pi_0(|\mathcal{F}^{\mathrm{red}}|)$ is a bijection between sets of connected components.*

Proof. Let $U, V \subseteq |\mathscr{D}|$ be two non-empty closed-open subsets with $V \cup U = |\mathscr{D}|$. Since $\mathrm{sp}_\mathscr{D}$ is a quotient map the map of connected components is surjective. Suppose now that $\emptyset \neq \mathrm{sp}_\mathcal{F}(U) \cap \mathrm{sp}_\mathcal{F}(V)$ we want to show that $U \cap V \neq 0$ which implies that $|\mathscr{D}|$ and $|\mathcal{F}^{\mathrm{red}}|$ have the same families of closed-open subsets. Since $\mathrm{sp}_\mathscr{D}$ is specializing we can assume there is a closed point $x \in \mathrm{sp}_{\mathscr{D}}(U) \cap \mathrm{sp}_{\mathscr{D}}(V)$. Since $|\mathcal{F}^{\mathrm{red}}|$ is locally Noetherian the closed points are open in the constructible topology. By proposition 1.4.33, $\widehat{\mathscr{D}}_{/x}$ is dense in ${\mathrm{sp}_{\mathscr{D}}}^{-1}(x)$, this implies that ${\mathrm{sp}_{\mathscr{D}}}^{-1}(x)$ is connected. Connectedness gives that $({\mathrm{sp}_{\mathscr{D}}}^{-1}(x) \cap U) \cap ({\mathrm{sp}_{\mathscr{D}}}^{-1}(x) \cap V) \neq \emptyset$ and in particular $U \cap V \neq \emptyset$ which is what we wanted to show. □

The following three technical lemmas will be used later on to prove that certain kimberlites are rich kimberlites.

Lemma 1.4.43. *Suppose $\mathcal{F}$ is a p-smelted kimberlite and that $\mathcal{F}_\eta$ is partially proper over $\mathbb{Q}_p^\diamond$, then:*

1. $\mathrm{sp}_{\mathcal{F}_\eta} : |\mathcal{F}_\eta| \to |\mathcal{F}^{\mathrm{red}}|$ *is specializing.*
2. *If $\mathrm{sp}_{\mathcal{F}_\eta}$ is surjective and $|\mathcal{F}^{\mathrm{red}}|$ is a locally Noetherian topological space then it is also a quotient map.*

Proof. Take a point $r \in |\mathcal{F}_\eta|$ mapping to $x \in |\mathcal{F}^{\mathrm{red}}|$ and take $y \in |\mathcal{F}^{\mathrm{red}}|$ specializing from x. We need to find q specializing from r that maps to y. Suppose r is represented by a map $f_r : \mathrm{Spa}(C, C^+) \to \mathcal{F}$ and suppose that $\mathcal{F}$ formalizes f_r. Let $K = O_C/C^{\circ\circ}$ and $K^+ = C^+/C^{\circ\circ}$, then x is the image of the maximal ideal of K^+ under the map $f_x : \mathrm{Spec}(K^+) \to \mathcal{F}^{\mathrm{red}}$. Consider the local ring R, constructed from $\mathcal{F}^{\mathrm{red}}$ by taking the reduced subscheme whose underlying topological spaces is the intersection of the closure of x and the localization at y. We let $k = K^+/\mathfrak{m}_{K^+}$, and so we have $R \subseteq k$. By ([56] Tag 00IA), we have a valuation subring $R \subseteq V \subseteq k$ such that $Frac(V) = k$ and V dominates R. This induces a valuation subring $K'^+ \subseteq K^+$ and a map $f_y : \mathrm{Spec}(K'^+) \to \mathcal{F}^{\mathrm{red}}$ whose closed point maps to y. In turn, this induces a valuation subring $C'^+ \subseteq C^+$ with $C'^+/C^{\circ\circ} = K'^+$ by lemma 1.4.4. Since $\mathcal{F}_\eta$ is partially proper, we get a map $f_q : \mathrm{Spa}(C, C'^+) \to \mathcal{F}_\eta$ extending f_r. Separatedness of $\mathcal{F}^{\mathrm{red}}$ can be used to prove that the point $q = [f_q] \in |\mathcal{F}_\eta|$ maps to y.

For the second claim, we first prove the case in which $|\mathcal{F}^{\mathrm{red}}|$ is irreducible. Let g be the generic point of $|\mathcal{F}^{\mathrm{red}}|$, and take a rank 1 point in $r \in |\mathcal{F}_\eta|$ mapping to g. Take a map $f_r : \mathrm{Spa}(C, O_C) \to \mathcal{F}_\eta$ representing r, and let C^{min} be the minimal integrally closed subring of C containing $\mathbb{Z}_p$ and $C^{\circ\circ}$, this is the minimal ring of integral elements for C. By partial properness we get a map $\mathrm{Spa}(C, C^{min}) \to \mathcal{F}_\eta$ whose image consists of the set of specializations of x in $|\mathcal{F}_\eta|$. The composition of the map $f^{min} : |\mathrm{Spa}(C, C^{min})| \to |\mathcal{F}^{\mathrm{red}}|$ is specializing, surjective and a spectral map of spectral spaces (surjectivity of this map proves that $|\mathcal{F}^{\mathrm{red}}|$ is also spectral instead of just locally spectral). By corollary 1.1.23 f^{min} is a closed map and consequenlty a quotient map of topological spaces.

The case in which $|\mathcal{F}^{\mathrm{red}}|$ has a finite number of irreducible components is analogous. For the general case, it is enough to prove locally on $|\mathcal{F}^{\mathrm{red}}|$ (for the Zariski topology) that $\mathrm{sp}_{\mathcal{F}_\eta}$ is a quotient map. By assumption around each point $x \in |\mathcal{F}^{\mathrm{red}}|$ there is an open neighborhood $U_x \subseteq |\mathcal{F}^{\mathrm{red}}|$ for which $|U_x|$ is a Noetherian topological space. In particular, U_x has a finite number of irreducible components and the closure $\overline{U}_x \subseteq |\mathcal{F}^{\mathrm{red}}|$ also has a finite number of irreducible components. Let $T = {\mathrm{sp}_{\mathcal{F}_\eta}}^{-1}(\overline{U}_x)$, this set is closed and consequently stable under specialization. If we lift the generic points of $\overline{U}_x$ to T we can argue as above to prove that the map $\mathrm{sp}_{\mathcal{F}_\eta} : T \to \overline{U}_x$ is a quotient map. This finishes the proof. □

Lemma 1.4.44. *Let C be a characteristic zero non-Archimedean algebraically closed field, and let $k = O_C/\mathfrak{m}_C$. Let $\mathcal{F}$ be a p-smelted kimberlite over $\mathrm{Spd}(O_C, O_C)$. Suppose that for every algebraically closed non-Archimedean field extension C'/C the basechange $\mathcal{F}_{O_{C'}} = \mathcal{F} \times_{\mathrm{Spd}(O_C,O_C)} \mathrm{Spd}(O_{C'}, O_{C'})$ satisfies that for closed point $x \in |\mathcal{F}^{\mathrm{red}}_{O_{C'}}|$ the p-adic tubular neighborhood $(\widehat{\mathcal{F}_{O_{C'}}}_{/x})_\eta$ is non-empty. Then $\mathrm{sp}_{\mathcal{F}_\eta}$ is a surjection.*

Proof. Given a point in $x \in |\mathcal{F}^{\mathrm{red}}|$ we can find a field extension of perfect fields K/k for which $\mathcal{F}^{\mathrm{red}} \times_k \mathrm{Spec}(K)$ has a section $y : \mathrm{Spec}(K) \to \mathcal{F}^{\mathrm{red}} \times_k \mathrm{Spec}(K)$ mapping to x under $\mathcal{F}^{\mathrm{red}} \times_k \mathrm{Spec}(K) \to \mathcal{F}^{\mathrm{red}}$. Since $\mathcal{F}$ is formally separated, $\mathcal{F}^{\mathrm{red}} \times_k \mathrm{Spec}(K)$ is also separated and sections to the structure map define closed points. We can construct a non-Archimedean field C' with $C \subseteq C'$ and $W(k)[\frac{1}{p}] \subseteq W(K)[\frac{1}{p}] \subseteq C'$. We get a map of p-smelted kimberlites $\mathcal{F}_{O_{C'}} \to \mathcal{F}$, and in $|\mathcal{F}^{\mathrm{red}}_{O_{C'}}|$ there is a closed point y mapping to x. Any point $r \in |\mathcal{F}_{C'}|$ with $\mathrm{sp}_{\mathcal{F}_{C'}}(r) = y$ maps to a point whose image under the specialization map is x. This proves surjectivity. $\square$

Lemma 1.4.45. *Let $f : \mathcal{F} \to \mathcal{G}$ be a map of p-adic kimberlites over $\mathbb{Z}_p^\diamondsuit$. Suppose that f is surjective, that $\mathcal{F}$ is a rich kimberlite, that $|\mathcal{G}^{\mathrm{red}}|$ is locally Noetherian and that $|f^{\mathrm{red}}|$ is a specializing map of topological spaces. Then the following hold:*

1. *$\mathcal{G}$ is rich.*
2. *If $\mathcal{F}$ has connected p-adic tubular neighborhoods and f^{red} has connected geometric fibers then $\mathcal{G}$ has connected p-adic tubular neighborhoods.*

Proof. Since the map $\mathcal{F}_\eta \to \mathcal{G}_\eta$ is surjective we have that, by proposition 1.4.34, $\mathcal{G}_\eta$ is a cJ-diamond. Since we assumed the kimberlites to be p-adic the map $\mathcal{F}^{\mathrm{red}} \to \mathcal{G}^{\mathrm{red}}$ is surjective, $|\mathcal{F}^{\mathrm{red}}| \to |\mathcal{G}^{\mathrm{red}}|$ is a quotient map by proposition 1.3.15 and a specializing map by hypothesis. Since we assumed that $|\mathcal{G}^{\mathrm{red}}|$ is locally Noetherian we only need to prove that $\mathrm{sp}_\mathcal{G}$ is specializing and a quotient map. Observe that the composition $|f^{\mathrm{red}}| \circ \mathrm{sp}_\mathcal{F}$ is specializing and quotient map, from which we can conclude.

For the claim on p-adic tubular neighborhoods pick a closed point $x \in |\mathcal{G}^{\mathrm{red}}|$, by proposition 1.4.26 $(\widehat{\mathcal{G}}_{/x})_\eta \times_\mathcal{G} \mathcal{F} = (\widehat{\mathcal{F}}_{/S})_\eta$ with $S = |f^{\mathrm{red}}|^{-1}(x)$. One can easily deduce from the hypothesis on geometric fibers that S is connected which implies by propositions 1.4.42 and 1.4.29 that $(\widehat{\mathcal{F}}_{/S})_\eta$ is also connected. Since f is surjective $(\widehat{\mathcal{G}}_{/x})_\eta$ is also connected. $\square$

Chapter 2

Specialization maps for some moduli problems

2.1 G-torsors, lattices and shtukas

In this section we recall the theory of vector bundles over the Fargues-Fontaine curve, and point to the technical statements that allow us to discuss the specialization map for the p-adic Beilinson-Drinfeld Grassmanians and moduli spaces of mixed-characteristic shtukas. Nothing in this section is essentially new and it is all written in some form in ([53], [30], [15], [1]). Nevertheless, we need specific formulations for some of these results that are not explicit in the literature. For the convenience of the reader, we justify how our formulations follow from other (harder) statements that can be explicitly found in the literature.

For the rest of this Chapter we let k be an auxiliary perfect field in characteristic p and we let $\mathscr{G}$ be a parahoric group scheme over $\mathrm{Spec}(W(k))$ with reductive generic fiber G. Depending on the context, we will introduce more notation and add restrictive hypothesis on what $\mathscr{G}$ and k are allowed to be. We will often times abbreviate $\mathrm{Spd}(W(k), W(k))$ by $W(k)^\diamond$, and $\mathrm{Spec}(k)^\diamond$ by $k^\diamond$.

2.1.1 Vector bundles, torsors and meromorphicity

We give a quick review of the theory of vector bundles for adic and perfectoid spaces. Given an analytic Huber pair (A, A^+), and an A module M we can define $\tilde{M}$ as a presheaf on the open sets of $\mathrm{Spa}(A, A^+)$ defined as $\tilde{M}(U) = \varprojlim_{\mathrm{Spa}(B,B^+)\subseteq U} M \otimes_A B$ running over all the rational subsets $\mathrm{Spa}(B, B^+) \subseteq \mathrm{Spa}(A, A^+)$, and where $M \otimes_A B$ refers to the usual tensor product of A-modules ignoring the topology (See [28] 1.3.2). Kedlaya and Liu prove that whenever (A, A^+) is sheafy and M is a finite projective A-module $\tilde{M}$ is an acyclic sheaf.

Definition 2.1.1. *Given an adic space X, a vector bundle of rank n over X is a sheaf of $\mathcal{O}_X$-modules which is locally isomorphic to $\tilde{M}(U_i)$ for some affinoid open cover $U_i = \mathrm{Spa}(A_i, {A_i}^+)$ and rank n projective modules $M(U_i)$ over A_i.*

In what follows we will need to work with categories of vector bundles over adic spaces and over schemes at the same time. The following important result of Kedlaya and Liu makes the bridge between these categories:

Theorem 2.1.2. *(See [28] 1.4.2) Let $X = \mathrm{Spa}(A, A^+)$ be an analytic affinoid adic space, suppose that A is sheafy. The functor*

$$H^0(\mathrm{Spa}(A, A^+), -) : Vec_{\mathrm{Spa}(A,A^+)} \to Vec_{\mathrm{Spec}(A)}$$

from the category of vector bundles over $\mathrm{Spa}(A, A^+)$ to the category of finite projective A-modules is an exact equivalence of exact categories.

Remark 2.1.3. *The acyclicity of $\tilde{M}$ proves that $H^0(\mathrm{Spa}(A, A^+), -)$ is exact. The quasi-inverse $\tilde{(-)}$ is also exact since $Tor_i(M, B) = 0$ for M finite projective A-module and $i > 0$.*

As in the theory of analytic functions on a complex variable one can introduce the notion of poles and meromorphic functions between vector bundles. We discuss how to do this:

Definition 2.1.4. *(See [53] 5.3.1, 5.3.2, 5.3.7) Given a uniform analytic adic space X, and an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$, we say that $\mathcal{I}$ defines a Cartier divisor if $\mathcal{I}$ is a line bundle over X. Let $Z \subseteq X$ denote the support of $\mathcal{O}_X/\mathcal{I}$. We say that $\mathcal{I}$ is a closed Cartier divisor if the topologically ringed topological space equipped with valuations $(Z, \mathcal{O}_X/\mathcal{I}, |\cdot|_{x\in Z})$ is an adic space.*

The data of a Cartier divisor allows us to define the notion of meromorphicity.

Proposition 2.1.5. *(See [53] 5.3.4) Let X be a uniform analytic adic space and $\mathcal{I} \subseteq \mathcal{O}_X$ a Cartier divisor. Let $U = X \setminus V(\mathcal{I})$ be the complement of the Cartier divisor and denote $j : U \subseteq X$ the natural inclusion, we then have inclusions of $\mathcal{O}_X$-modules:*

$$\mathcal{O}_X \subseteq \varinjlim \mathcal{I}^{\otimes(-n)} \subseteq j_*(\mathcal{O}_U)$$

Definition 2.1.6. *Let X be a uniform analytic adic space, let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two vector bundles over X and let $\mathcal{I} \subseteq \mathcal{O}_X$ be a Cartier divisor. Let us denote by U the complement of the support of $\mathcal{I}$. We say that a map in $Hom_U(\mathcal{V}_1, \mathcal{V}_2)$ is meromorphic along $\mathcal{I}$ if it is in*

$$H^0(X, \underline{Hom}(\mathcal{V}_1, \mathcal{V}_2) \otimes (\varinjlim \mathcal{I}^{\otimes(-n)}))$$

where $\underline{Hom}(\mathcal{V}_1, \mathcal{V}_2)$ denotes the internal Hom vector bundle.

Definition 2.1.7. *Let X be a uniform analytic adic space over $\mathrm{Spa}(W(k), W(k))$. We define a $\mathscr{G}$-torsor over X to be a $\otimes$-exact functor from the category of algebraic representations over finite free $W(k)$-modules, $Rep(\mathscr{G})$, to the category of vector bundles over X, Vec_X.*

We can then generalize the notion of meromorphicity to that of $\mathscr{G}$-torsors.

Definition 2.1.8. *With X as in definition 2.1.6 and definition 2.1.7, we will say that a morphism, $f : \mathscr{T}_{1|_U} \to \mathscr{T}_{2|_U}$, of $\mathscr{G}$-torsors over X, is meromorphic along $\mathcal{I}$ if for all representation $\pi \in Rep(\mathscr{G})$ the corresponding map $f(\pi) : \mathscr{T}_1(\pi)_{|_U} \to \mathscr{T}_2(\pi)_{|_U}$ is meromorphic along $\mathcal{I}$.*

We will often use the following fact.

Theorem 2.1.9. *(See [53] 17.1.8) The category of vector bundles fibered over* Perfd *forms a stack for the v-topology.*

2.1.2 Vector bundles on $\mathcal{Y}$

We defined $\mathbb{Z}_p^\diamondsuit$ as the v-sheaf parametrizing untilts. Although $\mathbb{Z}_p^\diamondsuit$ is not itself represented by an analytic adic space, the product $\mathbb{Z}_p^\diamondsuit \times S$ for any $S \in \text{Perf}$ can be represented by an analytic adic space. Let us recall this construction.

Definition 2.1.10. *Given a perfectoid Huber pair (R, R^+) and a pseudo-uniformizer $\varpi \in R^+$, we define $\mathcal{Y}^{R^+}_{[0,\infty)}$ as $\text{Spa}(W(R^+), W(R^+)) \setminus V([\varpi])$. Where $[\varpi]$ denotes a Teichműller lift of ϖ, and where $W(R^+)$ is given the $(p, [\varpi])$-adic topology. We also define $\mathcal{Y}_{R^+}$ as $\text{Spa}(W(R^+), W(R^+)) \setminus V(p, [\varpi])$.*

Proposition 2.1.11. *(See [29] 3.6, [53] 11.2.1]) For any perfectoid Huber pair (R, R^+) the space $\mathcal{Y}_{R^+}$ has a cover by sheafy Huber pairs. Consequently, $\mathcal{Y}_{R^+}$ and $\mathcal{Y}^{R^+}_{[0,\infty)}$ are adic spaces. Moreover, $(\mathcal{Y}^{R^+}_{[0,\infty)})^\diamondsuit = \mathbb{Z}_p^\diamondsuit \times \text{Spa}(R, R^+)$.*

Let us review the geometry of $\mathcal{Y}_{R^+}$, for this fix a pseudo-uniformizer $\varpi \in R^+$. One defines a continuous map $\kappa_\varpi : |\mathcal{Y}_{R^+}| \to [0, \infty]$ characterized by the property that $\kappa(y) = r$ if and only if for any positive rational number $r \leq \frac{m}{n}$ the inequality $|p|_y^m \leq |[\varpi]|_y^n$ holds and for any positive rational number $\frac{m}{n} \leq r$ the inequality $|[\varpi]|_y^n \leq |p|_y^m$ holds.

Given an interval $I \subseteq [0, \infty]$ we denote by $\mathcal{Y}^{R^+}_I$ the open subset corresponding to the interior of $\kappa_\varpi^{-1}(I)$. For example, $\mathcal{Y}^{R^+}_{(0,\infty]}$ corresponds to the locus in $\mathcal{Y}_{R^+}$ where $|p| \neq 0$ and $\mathcal{Y}^{R^+}_{[0,\infty)}$ corresponds to the locus where $|[\varpi]| \neq 0$. For intervals of the form $[0, \frac{h}{d}]$ where h and d are integers the space $\mathcal{Y}^{R^+}_{[0,\frac{h}{d}]}$ is represented by $\text{Spa}(R', R'^+)$ corresponding to the rational localization,

$$\{x \in \text{Spa}(W(R^+), W(R^+)) \mid |p^h|_x \leq |[\varpi]^d|_x \neq 0\}.$$

In this case, we can compute R'^+ explicitly as the $[\varpi]$-adic completion of $W(R^+)[\frac{p^h}{[\varpi]^d}]$ and R' as $R'^+[\frac{1}{[\varpi]}]$. A direct computation shows that R' does not depend of R^+. In particular, the exact category of vector bundles over $\mathcal{Y}^{R^+}_{[0,\infty)}$ does not depend of the choice of R^+ either.

We will also need to work with an algebraic version of $\mathcal{Y}_{R^+}$, which we will denote Y_{R^+}. This is defined as the scheme $\text{Spec}(W(R^+)) \setminus V(p, [\varpi])$. Since $W(R^+) \subseteq \mathcal{O}_{\mathcal{Y}_{R^+}}$ and since p, $[\varpi]$, do not vanish simultaneously on $\mathcal{Y}_{R^+}$ we get a map of locally ringed spaces $f : \mathcal{Y}_{R^+} \to Y_{R^+} \subseteq \text{Spec}(W(R^+)$.

Recall that given an untilt $R^\sharp$ of R there is a canonical surjection $W(R^+) \to R^{\sharp+}$ whose kernel is generated by an element $\xi \in W(R^+)$ primitive of degree 1 (See [53] 6.2.8). The element ξ defines a closed Cartier divisor over $\mathcal{Y}_{R^+}$ and also defines a Cartier divisor on the scheme Y_{R^+}. In what follows, we compare the categories of vector bundles over $\mathrm{Spec}(W(R^+))$, Y_{R^+} and $\mathcal{Y}_{R^+}$ with morphisms being functions that are meromorphic along the ideal (ξ).

Recall GAGA-type theorem of Kedlaya and Liu:

Theorem 2.1.12. *(See [29] 3.8) Suppose (R, R^+) is a perfectoid Huber pair in characteristic p. The natural morphisms of locally ringed spaces $f : \mathcal{Y}_{R^+} \to Y_{R^+}$ gives, via the pullback functor $f^* : Vec_{Y_{R^+}} \to Vec_{\mathcal{Y}_{R^+}}$, an exact equivalence of exact categories.*

Remark 2.1.13. *Although the reference does not explicitly claim that this equivalence is exact, one can simply follow the proof loc. cit. exchanging the word "equivalence" by "exact equivalence" since every arrow involved in the proof is an exact functor.*

Corollary 2.1.14. *With the notation as above, the pullback f^* induces an equivalence*

$$f^* : (Vec_{Y^{\xi\neq 0}_{R^+}})^{mer} \to (Vec_{\mathcal{Y}^{\xi\neq 0}_{R^+}})^{mer}$$

between the category whose objects are vector bundles over $\mathcal{Y}_{R^+}$ (respectively vector bundles over Y_{R^+}) and morphisms are functions meromorphic along the ideal (ξ) (respectively functions over $Y_{R^+} \setminus V(\xi)$).

Proof. By theorem 2.1.12 it is enough to prove that f^* is fully-faithful. Using internal $\underline{Hom}$ we can reduce to proving $H^0(\mathcal{Y}_{R^+}, f^*\mathcal{V})^{mer}_{\xi\neq 0} = H^0(Y^{\xi\neq 0}_{R^+}, \mathcal{V})$. For quasi-compact, quasi-separated schemes the global sections of a quasi-coherent sheaf after localizing by a global section of the structure sheaf is given simply by localization. That is $H^0(Y^{\xi\neq 0}_{R^+}, \mathcal{V}) = H^0(Y_{R^+}, \mathcal{V})[\frac{1}{\xi}]$. On the other hand, by definition

$$H^0(\mathcal{Y}_{R^+}, f^*\mathcal{V})^{mer}_{\xi\neq 0} = H^0(\mathcal{Y}_{R^+}, f^*\mathcal{V} \otimes \varinjlim (\xi)^{\otimes(-n)}).$$

Now the ideal sheaf (ξ) is isomorphic to $\mathcal{O}_{\mathcal{Y}_{R^+}}$ since it is a principal Cartier divisor so we can view $f^*\mathcal{V} \otimes \varinjlim(\xi)$ as:

$$f^*\mathcal{V} \xrightarrow{\xi} f^*\mathcal{V} \xrightarrow{\xi} \cdots$$

And since H^0 commutes with filtered colimits we get precisely $H^0(\mathcal{Y}_{R^+}, f^*\mathcal{V})[\frac{1}{\xi}]$. □

Since we defined $\mathscr{G}$-torsors Tannakianly these statements immediately generalize to those for $\mathscr{G}$-torsors. Kedlaya proves another important statement.

Theorem 2.1.15. *(See [29] 2.3, 2.7, 3.11) With notation as above, and letting j be the open embedding, $j : Y_{R^+} \to \mathrm{Spec}(W(R^+))$ the following statements hold:*

1. *The pullback functor $j^* : Vec_{\mathrm{Spec}(W(R^+))} \to Vec_{Y_{R^+}}$ is fully-faithful.*

2. *If R^+ is a valuation ring then j^* is an equivalence.*

3. *Taking categories of quasi-coherent sheaves the adjunction morphism $j^* j_* \mathcal{V} \to \mathcal{V}$ is an isomorphism.*

Remark 2.1.16. *One may think that the third statement together with the first statement of theorem 2.1.15 above would give an equivalence of categories of vector bundles for any ring R^+. This is not the case because even if $\mathcal{V}$ is a vector bundle, $j_*\mathcal{V}$ might not be a vector bundle over* $\mathrm{Spec}(W(R^+))$.

We will need a small modification of theorem 2.1.15.

For this we recall a few facts about topological modules on a Tate ring, this material is taken from ([53] 14.2.3). Let A be a complete Tate ring, f a topological nilpotent unit, A_0 a ring of definition, and N a projective module over A. One can endow N with its canonical topology (See [28] 1.1.11).

1. An A_0-submodule $M \subseteq N$ is open if and only if $M[\frac{1}{f}] = N$.

2. An A_0-submodule $M \subseteq N$ is bounded if and only if M is contained in a finitely generated A_0-submodule.

3. If $A \subseteq B$ with the subspace topology, B is Tate and complete for its topology, and $S \subseteq N \otimes_A B$ is bounded, then $S \cap N \subseteq N$ is also bounded.

The following statement is implicitly used and proved in ([53] 25.1.2).

Proposition 2.1.17. *Let $\mathrm{Spa}(R, R^+)$ be the product of points constructed from the family $\{(C_i, C_i{}^+), \varpi_i\}_{i\in I}$ as in definition 1.1.5. The pullback functor $j^* : Vec_{\mathrm{Spec}(W(R^+))} \to Y_{R^+}$ gives an equivalence of categories of vector bundles with fixed rank.*

Proof. We already have a fully-faithful embedding by theorem 2.1.15, so it is enough to prove it is essentially surjective. Let $\mathcal{V}$ be a vector bundle over Y_{R^+} of constant rank n, we let $M' = H^0(Y_{R^+}, \mathcal{V})$ which is a $W(R^+)$-module whose pullback to Y_{R^+} identifies with $\mathcal{V}$ by theorem 2.1.15, we want to prove that M' is a projective module. Let $N = M' \otimes_{W(R^+)} W(R^+)[\frac{1}{p}]$, this module is projective since N is the pullback of $\mathcal{V}$ to $\mathrm{Spec}(W(R^+)[\frac{1}{p}])$.

Define M_i as $H^0(Y_{C_i^+}, \iota_i^* \mathcal{V})$ where $\iota_i : Y_{C_i^+} \to Y_{R^+}$ is the closed embedding produced by the idempotent $1_i \in W(R^+) = \prod_{i\in I} W(C_i^+)$. For each i, this is a free $W(C_i^+)$-module by theorem 2.1.15 and because $W(C_i^+)$ is a local ring. We define $M = \prod_{i\in I} M_i$ which is a free $W(R^+)$-module of constant rank n. Since we have maps of $W(R^+)$ modules

$$M' = H^0(Y_{R^+}, \mathcal{V}) \to H^0(Y_{C_i^+}, \iota_i^* \mathcal{V}) = M_i$$

We get a map $M' \to M$. This map is injective since the family $Y_{C_i^+}$ is dense in the Zariski topology of Y_{R^+}. We want to prove that this map is an isomorphism.

As a first step we prove that the map induces an isomorphism $M'[\frac{1}{p}] \to M[\frac{1}{p}]$. For this we will consider $W(R^+)[\frac{1}{p}]$ as a Tate ring with its p-adic topology, and $W(R^+)$ as a ring of definition. In this context $M' \subseteq N$ is an open subset when N is given its canonical topology. This follows from property 1, since by construction $N = M'[\frac{1}{p}]$. The map of schemes $\mathrm{Spec}(W(R)) \to \mathrm{Spec}(W(R^+))$ factors through Y_{R^+}. This implies that $M' \otimes_{W(R^+)} W(R)$ is a finite projective module over $W(R)$. The usual map realizes $W(R^+)[\frac{1}{p}]$ as a topological subring of $W(R)[\frac{1}{p}]$. Moreover, $M' \otimes_{W(R^+)} W(R)$ is a bounded subset of $N \otimes_{W(R^+)[\frac{1}{p}]} W(R)[\frac{1}{p}]$ by property 2. On the other hand, property 3 readily implies M' is a bounded subset of N.

We construct an injection $M \subseteq N$ as $W(R^+)$-modules. Consider $N_i = 1_i N$ as a $W(C_i^+)[\frac{1}{p}]$-module but also as a subset of N. We have an injection $N \subseteq \prod_{i\in I} N_i$ and an element $(n_i)_{i\in I}$ is in the image of N if and only if the set $S = \{n_i\}_{i\in I} \subseteq N$ is bounded in N. There is a clear injection $M = \prod_{i\in I} M_i \to \prod_{i\in I} N_i$ and we claim that it factors through N. To prove the claim observe that if 1_i^c denotes the complementary idempotent of 1_i then $1_i \cdot M' = M'[\frac{1}{1_i^c}]$. Since taking global sections commutes with localization on qcqs schemes, we have that $M_i = 1_i \cdot M'$. Then the image of any element $m \in M$ in $\prod_{i\in I} N_i$ has the form $(m_i)_{i\in I}$ with $m_i \in 1_i \cdot M'$. Since M' is bounded in N, the set $\coprod_{i\in I} M_i$ is bounded and the map $M \to \prod_{i\in I} N_i$ defines and embedding into N. We have $M' \subseteq M \subseteq N$ and in particular $M'[\frac{1}{p}] = M[\frac{1}{p}]$, which finishes the first step.

We define $\mathcal{V}_2$ to be j^*M, which is a vector bundle over Y_{R^+}. The situation is as follows, we have a morphism of vector bundles $\mathcal{V}_1 \to \mathcal{V}_2$ over Y_{R^+} with $\mathcal{V}_2$ a trivial vector bundle, that becomes an isomorphism over $Y_{C_i^+}$ for every $i \in I$ and also becomes an isomorphism over $\mathrm{Spec}(W(R^+)[\frac{1}{p}]) \subseteq Y_{R^+}$. We prove that it is already an isomorphism over Y_{R^+}. After taking determinant bundles and fixing a trivialization we get a map $\wedge V_1 \to \mathcal{O}_X$, and it is enough to prove this one is an isomorphism.

Upon applying Beauville-Laszlo lemma (See [53] 5.2.9) to $p \in W(R^+)[\frac{1}{[\varpi]}]$ the morphism $\wedge\mathcal{V}_1 \to \mathcal{O}_X$ produces for us a family of lattices over $W(R) = (\widehat{W(R^+)[\frac{1}{[\varpi]}]})_p$ parametrized by $\mathrm{Spec}(R)$. This is the same as a morphism of schemes $\mathrm{Spec}(R) \to \mathrm{Gr}^{\mathbb{G}_m}_{\mathcal{W}}$ to the 1-dimensional Witt-vector Grassmanian (See [5] 8.1). Pullback of the map $\mathcal{V}_1 \to \mathcal{O}_X$ to $W(C_i)$ gives a lattice corresponding to the composition $\mathrm{Spec}(C_i) \to \mathrm{Spec}(R) \to \mathrm{Gr}^{\mathbb{G}_m}_{\mathcal{W}}$, but for each $i \in I$ the restriction of the morphism $\mathcal{V}_1 \to \mathcal{O}_X$ to $W(C_i)$ is an isomorphism. In particular, we get the following commutative diagram,

$$\begin{tikzcd} \coprod_{i\in I} \mathrm{Spec}(C_i) \arrow[r] \arrow[d] & \mathrm{Spec}(R) \arrow[d] \\ \mathrm{Spec}(\mathbb{F}_p) \arrow[r, "e"] & \mathrm{Gr}^{\mathbb{G}_m}_{\mathcal{W}} \end{tikzcd}$$

where the map $e : \mathrm{Spec}(\mathbb{F}_p) \to \mathrm{Gr}^{\mathbb{G}_m}_{\mathcal{W}}$ is the one associated to the identity of $\mathcal{O}_X$. The image of $|\coprod_{i\in I} \mathrm{Spec}(C_i)|$ in $|\mathrm{Spec}(R)|$ is dense since the map of rings $R \to \prod_{i\in I} C_i$ is injective. But $\mathrm{Gr}^{\mathbb{G}_m}_{\mathcal{W}}$ is representable by a discrete disjoint union of points of the form $\mathrm{Spec}(\mathbb{F}_p)$. So the map $\mathrm{Spec}(R) \to \mathrm{Gr}^{\mathbb{G}_m}_{\mathcal{W}}$ factors through the identity section which finishes the proof.

$\square$

Given $\xi \in W(R^+)$ primitive of degree 1 as before, observe that since both $\mathrm{Spec}(W(R^+))$

and Y_{R^+} are qcqs schemes the equivalence of vector bundles of proposition 2.1.17 generalizes to the categories where the objects are the same, but morphism are allowed to have poles along ξ on both categories.

Interestingly, extending $\mathscr{G}$-torsors from Y_{R^+} to $\mathrm{Spec}(W(R^+))$ adds yet another layer of complexity. Indeed, the equivalences of theorem 2.1.15 and proposition 2.1.17 are not exact equivalences, so Tannakian formalism can't be used directly. As a matter of fact, only the pullback functor j^* is exact. J. Anschűtz gives a detailed study of the problem of extending $\mathscr{G}$-torsors along j in [1]. We emphasize that, as we discussed in the introduction, the methods of [1] allow Anschűtz to construct a point-wise specialization map for the p-adic Beilinson-Drinfeld Grassmanians attached to any group $\mathscr{G}$ with parahoric reduction. Proposition 2.1.19 below, which is nothing but a small improvement to theorem 2.1.18 of Anschűtz, is the main technical input that we will need to upgrade Anschűtz map to a map of topological spaces. In the case that $\mathscr{G}$ is reductive we will be able to say more about the specialization map.

Theorem 2.1.18. *(See [1] 7.2, 7.3, 7.9, 6.5, 7.6 and [2] 11.6) Let C be an algebraically closed non-Archimedean field over k, let $C^+ \subseteq C$ an open and bounded valuation subring with $k \subseteq C^+$, and let $\mathscr{G}$ be a parahoric group scheme over $W(k)$. Then every $\mathscr{G}$-torsor $\mathscr{T}$ over Y_{C^+} extends to* $\mathrm{Spec}(W(C^+))$.

We now state an analogue of proposition 2.1.17 for $\mathscr{G}$-torsors.

Proposition 2.1.19. *Keep the notation as in theorem 2.1.18, and let* $\mathrm{Spa}(R, R^+)$ *be a product of points over k. Every $\mathscr{G}$-torsor $\mathscr{T}$ over Y_{R^+} extends along $j : Y_{R^+} \to \mathrm{Spec}(W(R^+))$ to a torsor $\mathscr{G}$ torsor over* $\mathrm{Spec}(W(R^+))$.

Proof. We need to prove that the functor $j_*\mathscr{T} : Rep(\mathscr{G}) \to Vec_{\mathrm{Spec}(W(R^+))}$ is exact, and since the functor j_* is always left-exact we only have to prove right-exactness of $j_*\mathscr{T}$. Suppose we have a morphism of free modules $f : \mathcal{V}_1 \to \mathcal{V}_2$ over $\mathrm{Spec}(W(R^+))$ and we have that the basechange to $\mathrm{Spec}(W(C_i^+))$ is surjective for every $i \in I$, we need to prove that the morphism is surjective. Taking determinant bundles we can reduce to the case that $\mathcal{V}_2$ is free of rank 1. After taking trivializations we have n sections $f_1, \cdots, f_n \in W(R^+)$ and we need to prove that they generate the unit ideal. Consider the family of subsets $\{I_m\}_{1\leq m\leq n}$ defined by

$$I_m = \{i \in I \mid f_m \in W(C_i^+)^\times\}$$

By construction 1_{I_m} is in the ideal generated by the f_i. Since each $W(C_i^+)$ is a local ring and the $\{f_m\}_{1\leq m\leq n}$ generate the unit ideal in $W(C_i^+)$ the union $\bigcup I_m$ has to be I. This finishes the proof. $\square$

We need the following descent result which is similar to theorem 2.1.9.

Proposition 2.1.20. *(See [53] 19.5.3) Let S be a perfectoid space over k and let $U \subseteq \mathcal{Y}^S_{[0,\infty)}$ be an open subset. For map of perfectoid spaces $f : S' \to S$, let $\mathcal{C}_{S'}$ denote the category of $\mathscr{G}$-torsors over $\mathcal{Y}^{S'}_{[0,\infty)} \times_{\mathcal{Y}^S_{[0,\infty)}} U$. Then the assignment $S' \mapsto \mathcal{C}_{S'}$, as a fibered category over* $\mathrm{Perf_S}$*, is a v-stack.*

2.1.3 Lattices and shtukas

For this section fix $\mathrm{Spa}(R,R^+)$ an affinoid perfectoid space over k, $\varpi \in R^+$ a choice of pseudo-uniformizer, $R^\sharp$ an untilt of R and $\xi_{R^\sharp}$ a generator for the kernel of the map $W(R^+) \to R^{\sharp,+}$.

Definition 2.1.21. *We define the category of $B^+_{dR}(R^\sharp)$-lattices with $\mathscr{G}$-structure to have as objects pairs $(\mathscr{T},\psi)$ where $\mathscr{T}$ is a $\mathscr{G}$-torsor over $\mathcal{Y}^{R^+}_{[0,\infty)}$ and $\psi : \mathscr{T} \to \mathscr{G}$ is an isomorphism over $\mathcal{Y}^{R^+}_{[0,\infty)} \setminus V(\xi_{R^\sharp})$ that is meromorphic along $(\xi_{R^\sharp})$. Morphisms are the evident isomorphisms of pairs.*

Given data $(\mathscr{T},\psi)$ as above we can choose a big enough number $r_\varpi \in \mathbb{R}$ for which $\mathcal{Y}^{R^+}_{[r_\varpi,\infty)}$ is disjoint from $V(\xi_{R^\sharp})$. Over this locus we can glue along ψ to extend $\mathscr{T}$ canonically to a $\mathscr{G}$-torsor over $\mathcal{Y}_{R^+}$. Using Corollary 2.1.14 and Beauville-Laszlo on the scheme $\mathrm{Spec}(W^+_R)[\frac{1}{[\varpi]}]$ we get an equivalence of categories with the category of pairs (Ξ,ψ) where Ξ is $\mathscr{G}$-torsor over $\mathrm{Spec}(B^+_{dR}(R^\sharp))$ and $\psi : \Xi \to \mathscr{G}$ is a trivialization over $\mathrm{Spec}(B_{dR}(R^\sharp))$, where $B^+_{dR}(R^\sharp)$ denotes the completion of $W(R^+)[\frac{1}{[\varpi]}]$ along $\xi_{R^\sharp}$, and $B_{dR}(R^\sharp) = B^+_{dR}(R^\sharp)[\frac{1}{\xi_{R^\sharp}}]$.

Recall that for an algebraically closed non-Archimedean field C the ring $B_{dR}(C^\sharp)$ is a complete discrete valuation field so that the set of isomorphism classes of $\mathscr{G}$-torsors over $\mathrm{Spec}(B^+_{dR}(C^\sharp))$ is in canonical bijection with $\mathscr{G}(B_{dR}(C^\sharp))/\mathscr{G}(B^+_{dR}(C^\sharp))$. In case $\frac{1}{p} \in C^\sharp$ we also have that $\frac{1}{p} \in B^+_{dR}(C^\sharp)$, and we will find that $\mathscr{G}_{B^+_{dR}(C^\sharp)} = G_{B^+_{dR}(C^\sharp)}$ is split reductive. After fixing auxiliary groups $T \subseteq B \subseteq G_{B^+_{dR}(C^\sharp)}$, a maximal torus and a Borel respectively, the Cartan decomposition gives an identification:

$$\mathscr{G}(B^+_{dR}(C^\sharp))\backslash\mathscr{G}(B_{dR}(C^\sharp))/\mathscr{G}(B^+_{dR}(C^\sharp)) = G(B^+_{dR}(C^\sharp))\backslash G(B_{dR}(C^\sharp))/G(B^+_{dR}(C^\sharp)) = X^+_*(T) \tag{2.1}$$

Suppose that B and T are fixed and understood from the context, and let $\mu \in X^+_*(T)$. We say that a $B^+_{dR}(C^\sharp)$-lattice (Ξ,ψ) is of type μ if the isomorphism class of (Ξ,ψ) maps to μ under the identification above.

We now consider mixed-characteristic shtukas. Recall that the spaces $\mathrm{Spec}(W(R^+))$, $\mathcal{Y}^{R^+}_{[0,\infty)}$, Y_{R^+} and $\mathcal{Y}_{R^+}$ come equipped with a Frobenious action. All of these actions are coming from the usual Frobenious action on $W(R^+)$ given by:

$$\phi^*\left(\sum_{i=0}^{\infty}[\alpha_i]p^i\right) = \sum_{i=0}^{\infty}[\alpha_i^p]p^i$$

A computation shows that $\phi(\mathcal{Y}^{R^+}_{[a,b]}) = \mathcal{Y}^{R^+}_{[pa,pb]}$ which proves that all of the loci considered above are preserved by Frobenious.

Definition 2.1.22. *We define the category of shtukas with one paw over* $\mathrm{Spa}(R^\sharp, {R^\sharp}^+)$ *and $\mathscr{G}$-structure. For this we require that $k = \mathbb{F}_p$ and that $\mathscr{G}$ is defined over $\mathbb{Z}_p$. This category has as objects pairs $(\mathscr{T},\Phi)$ where $\mathscr{T}$ is a $\mathscr{G}$-torsor over $\mathcal{Y}^{R^+}_{[0,\infty)}$ and $\Phi : \phi^*\mathscr{T} \to \mathscr{T}$ is an isomorphism over $\mathcal{Y}^{R^+}_{[0,\infty)} \setminus V(\xi_{R^\sharp})$ meromorphic along $(\xi_{R^\sharp})$. Morphisms are the evident isomorphisms of pairs.*

Definition 2.1.23. *Given a ϕ-module with $\mathscr{G}$-structure $(\mathcal{E}, \Phi_{\mathcal{E}})$ over $\mathcal{Y}^{R^+}_{(0,\infty)}$ and a shtuka $(\mathscr{T}, \Phi_{\mathscr{T}})$ we say that $(\mathscr{T}, \Phi_{\mathscr{T}})$ is isogenous to $(\mathcal{E}, \Phi_{\mathcal{E}})$ if there is a number $r \in \mathbb{R}$ (that depends of the choice of ϖ) and a ϕ-equivariant isomorphism $f : (\mathscr{T}, \Phi_{\mathscr{T}}) \to (\mathcal{E}, \Phi_{\mathcal{E}})$ defined over $\mathcal{Y}^{R^+}_{[r,\infty)}$. We call such a pair (r, f) an isogeny. Two isogenies (r_1, f_1) and (r_2, f_2) are equivalent if there is a third isogeny (r_3, f_3) with $r_3 > r_1, r_2$ and $f_1 = f_3 = f_2$ when restricted to $\mathcal{Y}^{R^+}_{[r_3,\infty)}$. We also refer by isogenies to the elements of the set of equivalence classes of pairs (r, f).*

After the work of Scholze and Weinstein one may think of mixed-characteristic shtukas as a generalization of p-divisible groups (See [53] 14.11,[52] Theorem B). We do not make this precise, but isogenies as defined above are closely related with isogenies of p-divisible groups. In what follows, we prove three technical lemmas that intuitively speaking allow us to "deform" lattices and shtukas with $\mathscr{G}$-structure. Later on it will become clear why we think of these lemmas as "deformation" statements.

For any $r \in [0, \infty)$ let $B^{R^+}_{[r,\infty]} = H^0(\mathcal{Y}^{R^+}_{[r,\infty]}, \mathcal{O}_{\mathcal{Y}^{R^+}_{[r,\infty]}})$, and consider $R^+_{\text{red}} = (R^+/\varpi)^{perf}$. We observe that the universal property of $\mathcal{Y}^{R^+}_{[r,\infty]}$ as a rational subset of $\operatorname{Spa}(W(R^+), W(R^+))$ induces compatible maps of rings $B^{R^+}_{[r,\infty]} \to W(R^+_{\text{red}})[\frac{1}{p}]$ for varying r. We denote this family of reduction maps by $(-_{\text{red}})$.

Lemma 2.1.24. *Let $s \in B^{R^+}_{[r,\infty]}$ and suppose that the reduction s_{red}, originally defined over $W(R^+_{\text{red}})[\frac{1}{p}]$, lies in $W(R^+_{\text{red}})$, then there are: a number r' with $r \leq r'$, elements $a \in W(R^+)$, $b \in B^{R^+}_{[r',\infty]}$ and a pseudo-uniformizer $\varpi_s \in R^+$ such that $s = a + b$ and $b \in [\varpi_s] \cdot B^{R^+}_{[r',\infty]}$.*

Proof. By enlarging r if necessary we can assume $\mathcal{Y}^{R^+}_{[r,\infty]}$ is of the form:

$$\{x \in \operatorname{Spa}(W(R^+), W(R^+)) \mid |[\varpi]|_x \leq |p^m|_x \neq 0\}$$

for some m, we compute $B^{R^+}_{[r,\infty]}$ explicitly. If S^+ denotes the p-adic completion of $W(R^+)[\frac{[\varpi]}{p^m}]$, then $B^{R^+}_{[r,\infty]} = S^+[\frac{1}{p}]$. Any element $s \in B^{R^+}_{[r,\infty]}$ is of the form $s = \frac{1}{p^n} \cdot \Sigma_{i=0}^{\infty}[a_i]x^{m(i)}p^i$ where $a_i \in R^+$, $x = \frac{[\varpi]}{p^m}$, and $m(i)$ denotes a non-negative integer. We can decompose $p^n \cdot s$ as

$$x \cdot \left(\sum_{i=0, m(i)>0}^{\infty} [a_i]x^{m(i)-1}p^i \right) + \sum_{i=0, m(i)=0}^{\infty} [a_i]p^i.$$

Since $x = \frac{[\varpi]}{p^m}$, we have that $[\varpi]$ divides in $B^{R^+}_{[r,\infty]}$ the first term of this decomposition. As long as we pick a ϖ_s that divides ϖ, we may and do reduce to the case $s = \Sigma_{i=0}^{\infty}[a_i]p^{i-n}$. In this case, $s_{\text{red}} = \Sigma_{i=0}^{\infty}[(a_i)_{\text{red}}]p^{i-n}$ and by hypothesis we have that for $i < n$ $(a_i)_{\text{red}} = 0$ in R^+_{red}. We can choose a pseudo-uniformizer ϖ_s for which all of a_i, for $i < n$, are zero in R^+/ϖ_s. We can take $a = \sum_{i=n}^{\infty}[a_i]p^{i-n}$ and $b = \sum_{i=0}^{n-1}[a_i]p^{i-n}$. These clearly satisfy the properties. $\square$

Lemma 2.1.25. *Suppose that $\mathscr{T}_1$ and $\mathscr{T}_2$ are trivial $\mathscr{G}$-torsor over* $\mathrm{Spec}(W(R^+))$ *and that* $\lambda : \mathscr{T}_1 \to \mathscr{T}_2$ *is an isomorphism defined over* $\mathcal{Y}^{R^+}_{[r,\infty]}$ *which upon reduction to* $\mathrm{Spec}(W(R^+_{\mathrm{red}})[\frac{1}{p}])$ *extends to an isomorphism over* $\mathrm{Spec}(W(R^+_{\mathrm{red}}))$*. Then, there is an isomorphism* $\widetilde{\lambda} : \mathscr{T}_1 \to \mathscr{T}_2$ *defined over* $\mathrm{Spec}(W(R^+))$*, a pseudo-uniformizer* $\varpi_\lambda \in R^+$ *and a number* $r' \in \mathbb{R}$ *with* $r \leq r'$ *such that* $\lambda = \widetilde{\lambda}$ *in:*

$$Hom_{\mathrm{Spec}(B^{R^+}_{[r',\infty]}/[\varpi_\lambda])}(\mathscr{T}_1, \mathscr{T}_2)$$

Proof. Fix for the rest of the proof trivializations $\iota_i : \mathscr{T}_i \to \mathscr{G}$, and consider $\iota_2 \circ \lambda \circ \iota_1^{-1}$ as an element $g \in H^0(\mathcal{Y}^{R^+}_{[r,\infty]}, \mathscr{G}) \subseteq H^0(\mathcal{Y}^{R^+}_{[r,\infty]}, GL_n)$ for some n and some embedding $\mathscr{G} \to GL_n$ defined over $W(k)$. By lemma 2.1.24 we can find ϖ_λ such that one can write g as $M_1 + [\varpi_\lambda]M_2$ where $M_1 \in GL_n(W(R^+))$ and $M_2 \in M_{n\times n}(B^{R^+}_{[r',\infty]})$. With this setup the reduction of M_1 to $GL_n(B^{R^+}_{[r',\infty]}/[\varpi_\lambda])$ lies in $\mathscr{G}(W(R^+)/[\varpi_\lambda])$. Moreover, since $\mathscr{G}$ is a smooth group and $W(R^+)$ is $[\varpi_\lambda]$-complete, we can lift this to an element $g' \in \mathscr{G}(W(R^+))$ with $g' = M_1$ in $GL_n(W(R^+)/[\varpi_\lambda])$. Consequently $g' = g$ in $\mathscr{G}(B^{R^+}_{[r',\infty]}/[\varpi_\lambda])$, and by letting $\widetilde{\lambda} = \iota_2^{-1} \circ g' \circ \iota_1$ we get the desired isomorphism. □

Remark 2.1.26. *In lemmas 2.1.25 and 2.1.24 above one can take $r = r'$ but that would extend the arguments and we will not need this.*

The proof of the following lemma is inspired by the computations that appear in [20] Theorem 5.6, and it is a key input in the proof of theorem 2.3.14.

Lemma 2.1.27 (Unique liftability of isogenies)**.** *Let $\mathscr{T}$ be a trivial $\mathscr{G}$-torsor defined over* $\mathrm{Spec}(W(R^+))$ *and let $\mathscr{G}_b$ denote the trivial $\mathscr{G}$-torsor endowed with the ϕ-module structure over* $\mathcal{Y}^{R^+}_{(0,\infty]}$ *given by an element* $b \in \mathscr{G}(\mathcal{Y}^{R^+}_{(0,\infty]})$*. Let* $\Phi : \phi^*\mathscr{T} \to \mathscr{T}$ *be an isomorphism defined over* $\mathrm{Spec}(W(R^+)[\frac{1}{\xi}])$ *and* $\lambda : \mathscr{T} \to \mathscr{G}_b$ *a ϕ-equivariant isomorphism defined over* $B^{R^+}_{[r,\infty]}/[\varpi]$ *for some r big enough so that $\xi_{R^\sharp}$ becomes a unit. Then, there is a unique ϕ-equivariant isomorphism* $\widetilde{\lambda} : \mathscr{T} \to \mathscr{G}_b$ *defined over* $\mathcal{Y}^{R^+}_{[r,\infty]}$ *such that* $\widetilde{\lambda} = \lambda$ *in* $B^{R^+}_{[r,\infty]}/[\varpi]$*.*

Proof. After fixing a trivialization $\iota : \mathscr{T} \to \mathscr{G}$ we may assume, by transport of structure, that $\mathscr{G} = \mathscr{T}$, that Φ is given by an element $\mathscr{G}(W(R^+)[\frac{1}{\xi}])$, and that λ is given by an element $\mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi])$. We need to find an element $\widetilde{\lambda} \in \mathscr{G}(B^{R^+}_{[r,\infty]})$ reducing to λ and satisfying $\Phi = \widetilde{\lambda}^{-1} \circ b \circ \phi^*(\widetilde{\lambda})$. Choose an arbitrary lift $\lambda_0 \in \mathscr{G}(B^{R^+}_{[r,\infty]})$ of λ, and let $\eta_0 = {\lambda_0}^{-1} \circ b \circ \phi^*(\lambda_0) \circ \Phi^{-1}$. We construct a pair of sequences of maps, $\lambda_i : \mathscr{G} \to \mathscr{G}_b$ and $\eta_i : \mathscr{G} \to \mathscr{G}$ defined recursively as follows:

$$\lambda_{n+1} = \lambda_n \circ \eta_n$$

$$\eta_n = \lambda_n^{-1} \circ b \circ \phi^*(\lambda_n) \circ \Phi^{-1}$$

We make the observation that $\eta_0 = Id$ in $\mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi])$ and we prove inductively that $\eta_n = Id$ in $\mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi^{p^n}])$. If $g \in \mathscr{G}(B^{R^+}_{[r,\infty]})$ is such that $g = Id$ in $\mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi^{p^n}])$, then $\phi^*(g) = Id$ in

$$\mathscr{G}(B^{R^+}_{[\frac{r}{p},\infty]}/[\varpi^{p^{n+1}}]) \subseteq \mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi^{p^{n+1}}]).$$

The induction then follows from the computation:

$$\begin{aligned}
\eta_{n+1} &= \lambda_{n+1}^{-1} \circ b \circ \phi^*(\lambda_{n+1}) \circ \Phi^{-1} && (2.2)\\
&= \eta_n^{-1} \circ \lambda_n^{-1} \circ b \circ \phi^*(\lambda_{n+1}) \circ \Phi^{-1} && (2.3)\\
&= \Phi \circ \phi^*(\lambda_n)^{-1} \circ b^{-1} \circ \lambda_n \circ \lambda_n^{-1} \circ b \circ \phi^*(\lambda_{n+1}) \circ \Phi^{-1} && (2.4)\\
&= \Phi \circ \phi^*(\lambda_n)^{-1} \circ \phi^*(\lambda_{n+1}) \circ \Phi^{-1} && (2.5)\\
&= \Phi \circ \phi^*(\eta_n) \circ \Phi^{-1} && (2.6)
\end{aligned}$$

Since $\phi^*(\eta_n) = Id$ in $\mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi^{p^{n+1}}])$ we also have that $\eta_{n+1} = Id$ in $\mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi^{p^{n+1}}])$.

This let us conclude that η_i converges to Id in $\mathscr{G}(B^{R^+}_{[r,\infty]})$. We define $\widetilde{\lambda} \in \mathscr{G}(B^{R^+}_{[r,\infty]})$ as the limit of the λ_i. Taking limits we deduce the identities $Id = \eta_\infty = \widetilde{\lambda} \circ b \circ \phi^*(\widetilde{\lambda}) \circ \Phi^{-1}$ and $\widetilde{\lambda} = \lambda_0 = \lambda$ in $\mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi])$ as we needed to show.

Suppose that there are two lifts $\widetilde{\lambda}_i$ of λ with the required properties. We get a ϕ-equivariant automorphism $\widetilde{\lambda}_1 \circ \widetilde{\lambda}_2^{-1}$ of $\mathscr{G}_b$ which we may think of as an element of $g \in \mathscr{G}(B^{R^+}_{[r,\infty]})$ that reduces to the identity in $B^{R^+}_{[r,\infty]}/[\varpi]$. Now, ϕ-equivariance gives $b = g^{-1} \circ b \circ \phi^*(g)$, and since $g = Id$ in $\mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi])$ then $\phi^*(g) = Id$ in $\mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi^p])$ and we get the identity $b = g^{-1} \circ b \circ Id$ in $\mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi^p])$. We may proceed inductively to prove that $g = Id$ in $\mathscr{G}(B^{R^+}_{[r,\infty]}/[\varpi^{p^n}])$ for every n. Since $B^{R^+}_{[r,\infty]}$ is complete and separated for the $[\varpi]$-adic topology we conclude that $g = Id$ in $\mathscr{G}(B^{R^+}_{[r,\infty]})$. $\square$

2.2 The specialization map for p-adic Beilinson-Drinfeld Grassmanians

2.2.1 Grassmanians as kimberlites

In the Berkeley notes, Scholze and Weinstein define a p-adic analogue of the Beilinson-Drinfeld Grassmanian where the parameter "curve" is given by $\mathbb{Z}_p^\Diamond$, or in our case $W(k)^\Diamond = \mathrm{Spec}(k)^\Diamond \times \mathbb{Z}_p^\Diamond$. We will adopt the definition that is the most convenient for studying the specialization map for this object.

Definition 2.2.1. *(See [53] 20.3.1) We let $\mathrm{Gr}^{\mathscr{G}}_{W(k)^\Diamond}$ denote the presheaf that assigns to an affinoid perfectoid pair (R, R^+) the set:*

$$\mathrm{Gr}^{\mathscr{G}}_{W(k)^\Diamond}(R, R^+) = \{(R^\sharp, \iota, f, \mathscr{T}, \psi)\}/\cong$$

Where $(R^\sharp, \iota, f)$ is an untilt over $W(k)$ and $(\mathscr{T}, \psi)$ is a lattice with $\mathscr{G}$-structure as in definition 2.1.21.

Whenever $\mathscr{G}$ is reductive over $W(k)$ with quasi-split fibers we fix $\mathfrak{T} \subseteq \mathfrak{B} \subseteq \mathscr{G}$, integrally defined maximal torus and Borel subgroups respectively.

Definition 2.2.2. *Suppose that $\mathscr{G}$ is reductive and $\mu \in X_*^+(\mathfrak{T})$ is a dominant cocharacter with reflex field E. We define $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E}$ as the subsheaf of $\mathrm{Gr}^{\mathscr{G}}_{O_E} := \mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond} \times_{W(k)^\diamond} O_E^\diamondsuit$ that on geometric points evaluates to $\mathscr{G}$-lattices (Ξ, ψ) whose type is bounded by μ in the Bruhat order. We use equation 2.1 to compare μ_Ξ with μ.*

Recall the following theorem of the Berkeley notes.

Theorem 2.2.3. *(See [53] 20.3.2, 21.2.1) For any parahoric group scheme $\mathscr{G}$ over $W(k)$, the presheaf $\mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}$ is a small v-sheaf and ind-proper over $W(k)^\diamondsuit$. Moreover, if $\mathscr{G}$ is reductive and $\mu \in X_*(\mathfrak{T})$ the functor $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E}$ is proper and representable in spatial diamonds over $O_E^\diamondsuit$. The inclusion of sheaves $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E} \to \mathrm{Gr}^{\mathscr{G}}_{O_E}$ is a closed embedding.*

Proposition 2.2.4. *The v-sheaf $\mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}$ formalizes products of points. In particular, it is v-formalizing.*

Proof. Let $\mathrm{Spa}(R, R^+)$ be a product of points and $f : \mathrm{Spa}(R, R^+) \to \mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}$ a map. By definition, associated to this map we have an untilt $(R^\sharp, \iota, m)$ over $W(k)$ and a $\mathscr{G}$-torsor $\mathscr{T}$ over $\mathcal{Y}^{R^+}_{[0,\infty)}$ together with a trivialization $\psi : \mathscr{T} \to \mathscr{G}$ over $\mathcal{Y}^{R^+}_{[0,\infty)} \setminus V(\xi_{R^\sharp})$ meromorphic along $\xi_{R^\sharp}$. We can use ψ to glue $\mathscr{T}$ and $\mathscr{G}$ along $\mathcal{Y}^{R^+}_{[r,\infty)}$ (for big enough r) and get a $\mathscr{G}$-torsor defined over $\mathcal{Y}_{R^+}$, together with a meromorphic isomorphism over $\mathcal{Y}_{R^+} \setminus V(\xi_{R^\sharp})$ which restricts to the original data. Using corollary 2.1.14, proposition 2.1.19 and the fact that by construction $\mathscr{T}$ is trivial on $Y_{R^+} \setminus V(\xi)$ we can extend $\mathscr{T}$ to a $\mathscr{G}$-torsor over $\mathrm{Spec}(W(R^+))$ together with a trivialization over $\mathrm{Spec}(W(R^+)[\frac{1}{\xi_{R^\sharp}}])$. We claim that this is enough to define a map $\mathrm{Spd}(R^+, R^+) \to \mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}$. Indeed, take a second affinoid perfectoid $\mathrm{Spa}(S, S^+)$ and a map $g : \mathrm{Spa}(S, S^+) \to \mathrm{Spd}(R^+, R^+)$, we want to produce a map $\mathrm{Spa}(S, S^+) \to \mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}$ in a functorial way. We may construct an untilt $(S^\sharp, \iota, m)$ as in lemma 1.4.8. The map g gives a map $g' : W(R^+) \to W(S^+)$ with $g'(\xi_{R^\sharp}) = \xi_{S^\sharp}$. Basechange along g' gives a $\mathscr{G}$-torsor over $\mathrm{Spec}(W(S^+))$ together with a trivialization over $\mathrm{Spec}(W(S^+)[\frac{1}{g'(\xi)}])$. This restricts to a $\mathscr{G}$-torsor over $\mathcal{Y}^{S^+}_{[0,\infty)}$ and a trivialization over $\mathcal{Y}^{S^+}_{[0,\infty)} \setminus V(g'(\xi))$ that is meromorphic along $g'(\xi)$. This gives our desired natural transformation $\mathrm{Spd}(R^+, R^+) \to \mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}$. Clearly the composition $\mathrm{Spa}(R, R^+) \to \mathrm{Spd}(R^+, R^+) \to \mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}$ agrees with f, so this map is a formalization. $\square$

Proposition 2.2.5. *(See [53] Section 20.3) The v-sheaf $\mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}$ is specializing and formally p-adic, and $(\mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond})^{red}$ is represented by the Witt-vector Grassmanian, $\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k}$. Moreover, if $\mathscr{G}$ is reductive and k_E denotes the residue field of O_E, then $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\diamondsuit}$ is also formally p-adic and $(\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\diamondsuit})^{\mathrm{red}} = \mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_E}$.*

Proof. That $\mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}$ is specializing would follow from proposition 2.2.4, theorem 2.2.3 and proposition 1.3.31 once we establish that it is formally p-adic. We begin by constructing a map $\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k} \to (\mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond})^{red}$. Given a map $\mathrm{Spec}(R) \to \mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k}$ we need to produce a map $\mathrm{Spec}(R)^\diamond \to \mathrm{Gr}^{\mathscr{G}}_{W(k)^\diamond}$ in a functorial way. The map to $\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k}$ is given by a $\mathscr{G}$-torsor $\mathscr{T}$

over $\operatorname{Spec}(W(R))$ together with a trivialization $\psi : \mathscr{T} \to \mathscr{G}$ over $\operatorname{Spec}(W(R)[\frac{1}{p}])$. Given an affinoid perfectoid $\operatorname{Spa}(S, S^+)$ and a map $f : \operatorname{Spa}(S, S^+) \to \operatorname{Spec}(R)^\diamond$ we need to produce a map $\operatorname{Spa}(S, S^+) \to \operatorname{Gr}^{\mathscr{G}}_{W(k)^\diamond}$. The morphism f induces the ring map $f' : W(R) \to W(S^+)$. We can assign to f the characteristic p untilt and assign the $\mathscr{G}$-bundle $f'^*\mathscr{T}$ over $\mathcal{Y}^{S^+}_{[0,\infty)}$ with trivialization $f'^*\psi$, using corollary 2.1.14 we see that it is meromorphic along p. This construction is clearly functorial and gives the desired map.

Now, by Beauville-Laszlo theorem, we may also think of $(f'^*\mathscr{T}, f'^*\psi)$ as a pair (Ξ_S, ψ_S) with Ξ_S a $\mathscr{G}$-torsor over $\operatorname{Spec}(B^+_{dR}(S))$ and $\psi_S : \Xi_S \to \mathscr{G}$ a trivialization over $\operatorname{Spec}(B_{dR}(S))$. Since S is the characteristic p untilt, we have $B^+_{dR}(S) = W(S)$ and $B_{dR}(S) = W(S)[\frac{1}{p}]$. In this case, (Ξ_S, ψ_S) is simply the pullback of $(\mathscr{T}, \psi)$ along $W(R) \to W(S)$. In particular, if $\mathscr{G}$ is reductive and the type of $(\mathscr{T}, \psi)$ is pointwise bounded by $\mu \in X^+_*(\mathfrak{T})$, then the type of (Ξ_S, ψ_S) is also pointwise bounded by μ. This last observation gives us a commutative diagram which we will use later in the proof:

$$\begin{tikzcd}
(\operatorname{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_E})^\diamond \arrow[r] \arrow[d] & (\operatorname{Gr}^{\mathscr{G}}_{\mathcal{W},k_E})^\diamond \arrow[d] \\
\operatorname{Gr}^{\mathscr{G},\leq\mu}_{O_E^\diamond} \arrow[r] & \operatorname{Gr}^{\mathscr{G}}_{O_E^\diamond}
\end{tikzcd} \tag{2.7}$$

For the moment, let us move on and prove explicitly that for any (R, R^+) we have bijection of sets:
$$(\operatorname{Gr}^{\mathscr{G}}_{\mathcal{W},k})^\diamond(R, R^+) \to \operatorname{Gr}^{\mathscr{G}}_{W(k)^\diamond} \times_{W(k)^\diamond} \operatorname{Spec}(k)^\diamond(R, R^+).$$
By lemma 1.3.35, this would give that $\operatorname{Gr}^{\mathscr{G}}_{W(k)^\diamond} \to W(k)^\diamond$ is formally adic and would prove $(\operatorname{Gr}^{\mathscr{G}}_{W(k)^\diamond})^{\mathrm{red}} = \operatorname{Gr}^{\mathscr{G}}_{\mathcal{W},k}$. To prove injectivity, suppose we are given two maps $g_i : \operatorname{Spa}(R, R^+) \to (\operatorname{Gr}^{\mathscr{G}}_{\mathcal{W},k})^\diamond$ in characteristic p whose composition agree. It is enough to prove that $g_1 = g_2$ after taking a v-cover of $\operatorname{Spa}(R, R^+)$. Locally for the v-topology we can assume that both maps factor through morphisms $g_i' : \operatorname{Spec}(R^+) \to \operatorname{Gr}^{\mathscr{G}}_{\mathcal{W},k}$ given by pairs $(\mathscr{T}_i, \psi_i)$. Since the compositions agree, these pairs become isomorphic over $\mathcal{Y}^{R^+}_{[0,\infty)}$. Since both $\mathscr{T}_i$ are defined over $\operatorname{Spec}(W(R^+))$ and the pullback functor $j^* : Vec_{\operatorname{Spec}(W(R^+))} \to Vec_{Y_{R^+}}$ of theorem 2.1.15 is fully faithful (even when it is not an equivalence), we can conclude that $g_1' = g_2'$.

To prove surjectivity take a map $f : \operatorname{Spa}(R, R^+) \to \operatorname{Gr}^{\mathscr{G}}_{W(k)^\diamond} \times_{W(k)^\diamond} \operatorname{Spec}(k)^\diamond$. Since surjectivity can be checked v-locally we can assume that $\operatorname{Spa}(R, R^+)$ is a product of points. By the proof of proposition 2.2.4 we get a $\mathscr{G}$-torsor $\mathscr{T}$ over $\operatorname{Spec}(W(R^+))$ and a trivialization over $\operatorname{Spec}(W(R^+)[\frac{1}{p}])$ which gives a map $\operatorname{Spec}(R^+) \to \operatorname{Gr}^{\mathscr{G}}_{\mathcal{W},k}$ and consequently the required lift to our original map $\operatorname{Spa}(R, R^+) \to (\operatorname{Gr}^{\mathscr{G}}_{\mathcal{W},k})^\diamond$.

The second claim will follow from proving that the commutative diagram 2.7 is Cartesian. Indeed, that would prove that the closed immersion $\operatorname{Gr}^{\mathscr{G},\leq\mu}_{O_E^\diamond} \to \operatorname{Gr}^{\mathscr{G}}_{O_E^\diamond}$ is formally adic, and that $(\operatorname{Gr}^{\mathscr{G},\leq\mu}_{O_E^\diamond})^{\mathrm{red}} = \operatorname{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_E}$. All of the morphisms in diagram 2.7 are closed immersions, so it is enough to check that the diagram is Cartesian on (C, O_C)-points. Suppose we have a map $m : \mathrm{Spa}(C, O_C) \to \mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\lozenge} \cap (\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k_E})^\lozenge$, this factors through a map $m' : \mathrm{Spec}(O_C)^\lozenge \to (\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k_E})^\lozenge$. The map is given by a lattice $(\mathscr{T}_{O_C}, \psi_{O_C})$ over $W(O_C)$ whose basechange to $W(C)$ is bounded by μ. Indeed, using the Beauville-Laszlo theorem, this is the translation of what it means for the composition $\mathrm{Spa}(C, O_C) \subseteq \mathrm{Spec}^\lozenge(O_C) \to (\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k_E})^\lozenge$ to factor through $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\lozenge}$. By proposition 1.3.17 m' is coming from a map $\mathrm{Spec}(O_C) \to \mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k_E}$ for which the composition $\mathrm{Spec}(C) \to \mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k_E}$ factors through $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_E}$. Since $\mathrm{Spec}(C)$ is Zariski dense in $\mathrm{Spec}(O_C)$, m' factors through $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_E}$ and m factors through $(\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_E})^\lozenge$. □

Remark 2.2.6. *We want to remark that although the v-sheaf $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\lozenge}$ is formally p-adic, the similarly defined moduli $\mathrm{Gr}^{\mathscr{G},\mu}_{O_E^\lozenge}$ of B^+_{dR}-lattices of type exactly μ is not formally p-adic if μ is not minuscule. Indeed, in that case there are points $p : \mathrm{Spa}(C, O_C) \to (\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k_E})^\lozenge$ with formalization $m : \mathrm{Spd}(O_C, O_C) \to (\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k_E})^\lozenge$ such that p factors through $\mathrm{Gr}^{\mathscr{G},\mu}_{O_E^\lozenge}$ but m doesn't. This implies that $\mathrm{Gr}^{\mathscr{G},\mu}_{O_E^\lozenge} \cap (\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k_E})^\lozenge$ is not v-formalizing and consequently not represented by a scheme-theoretic v-sheaf. Similarly, this proves that if μ is not minuscule $\mathrm{Gr}^{\mathscr{G},\mu}_{O_E^\lozenge} \cap (\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},k_E})^\lozenge$ properly contains $(\mathrm{Gr}^{\mathscr{G},\mu}_{\mathcal{W},k_E})^\lozenge$ as open subsheaves of $(\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_E})^\lozenge$.*

Corollary 2.2.7. *If $\mathscr{G}$ is reductive over $W(k)$ and $\mu \in X^+_*(\mathfrak{T})$, then the v-sheaf $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E}$ is a p-adic kimberlite.*

Proof. We know $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\lozenge}$ is separated by theorem 2.2.3. Since $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\lozenge}$ is formally p-adic, by proposition 1.3.31 it is also formally separated. The map $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\lozenge} \to \mathrm{Gr}^{\mathscr{G}}_{O_E^\lozenge}$ is a formally closed immersion and $\mathrm{Gr}^{\mathscr{G}}_{O_E^\lozenge}$ is specializing so by proposition 1.4.30 $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\lozenge}$ is also specializing. The morphism $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\lozenge} \to O_E^\lozenge$ is formally adic which implies that the adjunction morphism $((\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\lozenge})^{red})^\lozenge \to \mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\lozenge}$ is a closed embedding. By proposition 2.2.5 $(\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\lozenge})^{red} = \mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_E}$, which is represented by a scheme (See [5] Theorem 8.3). We also have $(\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\lozenge})^{an} = \mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E^\lozenge} \times_{O_E^\lozenge} E^\lozenge$, which is represented by a spatial diamond by theorem 2.2.3. □

The purpose of the rest of this section is to upgrade corollary 2.2.7 and prove that $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_E}$ is a rich p-adic kimberlite and that its p-adic tubular neighborhoods are connected.

Remark 2.2.8. *One could try to generalize corollary 2.2.7 to the parahoric case. This is more subtle to deal with because it is not possible to define boundedness conditions through a cocharacter. What one can do is to define the boundedness condition on the generic fiber and take the closure in the sense of v-sheaves to obtain a closed subsheaf. To prove that this closed subsheaf is a kimberlite the only real difficulty one can run into is that it is not a priori clear whether the special fiber of this subsheaf is represented by a scheme or not. It is the author's understanding that these subtleties will be tackled in [26], and that their methods can prove that the special fiber will be represented by the "expected" perfect scheme. With this* *result at hand one can prove that the bounded parahoric Beilinson-Drinfeld Grassmanians are also rich p-adic kimberlites.*

2.2.2 Twisted loop sheaves

We begin by discussing two constructions that are related to twisted loop sheaves and that we will use below. Given an affine scheme $X = \mathrm{Spec}(A)$ of finite type over $W(k)$ with structure morphism $\pi : X \to \mathrm{Spec}(W(k))$, we can associate to it two v-sheaves over $W(k)^\lozenge$ which we will denote by $X(\mathcal{O}^\sharp)$ and $X(\mathcal{O}^{\sharp,+})$. Here $X(\mathcal{O}^\sharp) : \mathrm{Perf_k} \to \mathrm{Sets}$ is defined to be the presheaf that assigns to $\mathrm{Spa}(R, R^+)$ the set of triples $(R^\sharp, \iota, f)$ where $(R^\sharp, \iota)$ is an untilt and $f \in Hom_{W(k)}(A, R^\sharp)$ is a $W(k)$-algebra homomorphism. On the other hand, $X(\mathcal{O}^{\sharp,+})$ assigns triples $(R^\sharp, \iota, f)$ with $f \in Hom_{W(k)}(A, R^{\sharp,+})$. Notice that we have an open inclusion of v-sheaves $X(\mathcal{O}^{\sharp,+}) \subseteq X(\mathcal{O}^\sharp)$. Both of these functors glue to give a construction that is now defined for every scheme X locally of finite type over $\mathrm{Spec}(W(k))$. Visibly, these two constructions are very related to the functor $\lozenge : \mathrm{PreAd}_{W(k)} \to \widetilde{\mathrm{Perf}}$, we make this explicit below.

We still assume $X = \mathrm{Spec}(A)$, and we let X_p denote the p-adic completion of X. Now, X_p is a p-adic Noetherian formal scheme that we may regard as an affinoid adic space $\mathrm{Spa}(A_p, A_p)$. Since for any untilt of R the ring $R^{\sharp,+}$ is p-adically complete, we have an identification $X_p^\lozenge = X(\mathcal{O}^{\sharp,+})$. Also, if $Y \to X$ is an open cover of the form $Y = \coprod_{i=1}^n \mathrm{Spec}(A[\frac{1}{f_i}])$ with $f_i \in A$, then $Y_p \to X_p$ is also an open cover of adic spaces. Indeed, $\mathrm{Spec}(A[\frac{1}{f_i}])_p$ corresponds to the open subset of X_p where $1 \leq |f_i|$.

The construction of $X(\mathcal{O}^\sharp)$ is a little more elaborate. Given an adic space S (thought of as a triple $(|S|, \mathcal{O}_S, \{v_s : s \in |S|\})$ in Huber's category $\mathscr{V}$ see [24]), we let S^H denote the topologically ringed space $(|S|, \mathcal{O}_S)$ that is obtained from S by forgetting the last entry of data. Suppose we are given a morphism of schemes $f : X \to Y$ that is locally of finite type and a morphism $g : S^H \to Y$ of locally ringed spaces where S is an adic space for which every point $s \in S$ has an affinoid open neighborhood with Noetherian ring of definition. In [24] (proposition 3.8) Huber constructs an adic space "$S \times_Y X$" together with a map of adic spaces $p_1 :$ "$S \times_Y X$" $\to S$ and a map of locally ringed spaces $p_2 : (\text{"}S \times_Y X\text{"})^H \to X$ with the following universal property. If T is an adic space, $\pi_1 : T \to S$ is a map of adic spaces and $\pi_2 : T^H \to X$ is a map of locally ringed spaces such that $f \circ \pi_1 = g \circ \pi_2^H$, then there is a unique map $\pi : T \to$ "$S \times_Y X$" such that $p_1 \circ \pi = \pi_1$ and $p_2 \circ \pi^H = \pi_2$.

With this adic space at hand we can let $Y = \mathrm{Spec}(W(k))$, $S = \mathrm{Spa}(W(k), W(k))$ and X the finite type scheme over Y that we started with and define X^{ad} as ("$S \times_Y X$"). With this definition we have $X(\mathcal{O}^\sharp) = (X^{ad})^\lozenge$. Moreover, if $X = \mathrm{Spec}(A)$ and $X_f = \mathrm{Spec}(A[\frac{1}{f}])$ for $f \in A$ we can see from the universal property that X_f^{ad} is the open locus of X^{ad} where $f \neq 0$.

The advantage of these two-step constructions is that it makes it clear, by proposition 1.1.30, that $X(\mathcal{O}^\sharp)$ and $X(\mathcal{O}^{\sharp,+})$ are small v-sheaves and it also clarifies the glueing process for $X(\mathcal{O}^\sharp)$ and $X(\mathcal{O}^{\sharp,+})$ when X is an arbitrary finite type scheme. These two constructions already appear in [51] §27.

We will later on use the following facts about these constructions:

Proposition 2.2.9. *If $X \to \mathrm{Spec}(W(k))$ is a proper map of schemes, then the natural map $X(\mathcal{O}^{\sharp,+}) \to X(\mathcal{O}^{\sharp})$ is an isomorphism.*

Proof. By [24] remark 4.6.(iv).d we have an isomorphism of adic spaces $X_p \to X^{ad}$ where X_p and X^{ad} are as above. Since $X(\mathcal{O}^{\sharp}) = (X^{ad})^{\lozenge}$ and $X(\mathcal{O}^{\sharp,+}) = X_p^{\lozenge}$ the conclusion holds. $\square$

Proposition 2.2.10. *Suppose that X and Y are qcqs finite type schemes over* $\mathrm{Spec}(W(k))$ *and that $X \to Y$ is universally subtrusive as in definition 1.3.4, then $X(\mathcal{O}^{\sharp}) \to Y(\mathcal{O}^{\sharp})$ and $X(\mathcal{O}^{\sharp,+}) \to Y(\mathcal{O}^{\sharp,+})$ are surjective maps of v-sheaves.*

Proof. Replacing Y by an open cover we may assume that $Y = \mathrm{Spec}(A)$ for a ring A of finite type over $W(k)$. By [47] theorem 3.12 we may assume that $X \to Y$ factors as $X \to Y' \to Y$ with $Y' \to Y$ proper and surjective and $X \to Y'$ a quasi-compact open covering. Since open covers of adic spaces induce surjective maps of v-sheaves we only need to deal with the proper case. Moreover, by Chow's lemma ([56] Tag 0200) we may assume $Y' \to Y$ is projective. We claim that both maps of v-sheaves $Y'(\mathcal{O}^{\sharp}) \to Y(\mathcal{O}^{\sharp})$ and $Y'(\mathcal{O}^{\sharp,+}) \to Y(\mathcal{O}^{\sharp,+})$ are quasi-compact. Indeed, they are both the composition of a closed immersion and the first projection map of $(\mathbb{P}^n_{W(k)})^{\lozenge} \times_{W(k)^{\lozenge}} Y(\mathcal{O}^{\sharp})$ and $(\mathbb{P}^n_{W(k)})^{\lozenge} \times_{W(k)^{\lozenge}} Y(\mathcal{O}^{\sharp,+})$ respectively. By ([51] 12.11) we may check surjectivity at a topological level. Take an algebraically closed non-Archimedean field C with open and bounded valuation subring $C^+ \subseteq C$, and consider ring maps $r^* : A \to C$ and $s^* : A \to C^+$ representing (C, C^+)-valued points in $r \in Y(\mathcal{O}^{\sharp})$ and $s \in Y(\mathcal{O}^{\sharp,+})$ respectively. Since $Y' \to Y$ is proper and surjective the map of schemes $\mathrm{Spec}(C) \times_Y Y' \to \mathrm{Spec}(C)$ admits a section which induces a lift of r to $Y'(\mathcal{O}^{\sharp})$. Analogously, the map of schemes $\mathrm{Spec}(C^+) \times_Y Y' \to \mathrm{Spec}(C^+)$ admits a section (by the valuative criterion of properness). This defines an element of $Y'(\mathcal{O}^{\sharp,+})$ lifting s.

$\square$

Perhaps unsurprisingly, for a scheme X over $\mathrm{Spec}(W(k))$ the reduction functor applied to $X(\mathcal{O}^{\sharp})$ and $X(\mathcal{O}^{\sharp,+})$ give the same scheme-theoretic v-sheaf.

Proposition 2.2.11. *Given X a scheme locally of finite type over* $\mathrm{Spec}(W(k))$ *we have identifications in* $\widetilde{\mathrm{SchPerf}}$*:*

$$X(\mathcal{O}^{\sharp})^{\mathrm{red}} \cong X \times_{W(k)} \mathrm{Spec}(k) \cong X(\mathcal{O}^{\sharp,+})^{\mathrm{red}}$$

Proof. Both identifications follow from proposition 1.3.20. By the construction of X_p as a p-adic completion in the case of $X(\mathcal{O}^{\sharp,+})$, and by the universal property of X^{ad} in the category of adic spaces in the case of $X(\mathcal{O}^{\sharp})$.

$\square$

We move on to discuss twisted loop sheaves. For the rest of this section we let C be an algebraically closed non-Archimedean field over k with ring of integers O_C and residue field k_C. We fix a characteristic 0 untilt $C^{\sharp}$ and we pick $\xi \in W(O_C)$ a generator for the kernel of $W(O_C) \to O_{C^{\sharp}}$. The choice of untilt determines a unique map $O_C^{\lozenge} \to \mathbb{Z}_p^{\lozenge}$ that we also fix throughout this section.

Definition 2.2.12. *1. We let* $W^+(\mathcal{O}) : \mathrm{Perf}_{O_C^\lozenge} \to \mathrm{Sets}$ *denote the presheaf that assigns to* $\mathrm{Spa}(R, R^+) \to O_C^\lozenge$ *the ring* $W(R^+)$*.*

2. *We let* $B^+_{dR}(\mathcal{O}^\sharp) : \mathrm{Perf}_{O_C^\lozenge} \to \mathrm{Sets}$ *denote the presheaf that assigns to* $\mathrm{Spa}(R, R^+) \to O_C^\lozenge$ *the ring* $B^+_{dR}(R^\sharp)$ *where* $R^\sharp$ *is the untilt associated with our fixed choice of* $\xi \in W(O_C)$*.*

3. *We let* $W(\mathcal{O}) : \mathrm{Perf}_{O_C^\lozenge} \to \mathrm{Sets}$ *denote the presheaf that assigns to* $\mathrm{Spa}(R, R^+) \to O_C^\lozenge$ *the ring* $W(R^+)[\frac{1}{\xi}]$*.*

4. *We let* $B_{dR}(\mathcal{O}^\sharp) : \mathrm{Perf}_{O_C^\lozenge} \to \mathrm{Sets}$ *denote the presheaf that assigns to* $\mathrm{Spa}(R, R^+) \to O_C^\lozenge$ *the ring* $B_{dR}(R^\sharp) := B^+_{dR}(R^\sharp)[\frac{1}{\xi}]$*.*

Proposition 2.2.13. *The presheaves* $W^+(\mathcal{O})$*,* $B^+_{dR}(\mathcal{O}^\sharp)$*,* $W(\mathcal{O})$*, and* $B_{dR}(\mathcal{O}^\sharp)$ *are small* v*-sheaves.*

Proof. Ignoring the ring structure, we see that the Teichműller expansion of $W(R^+)$ gives a bijection to $(R^+)^{\mathbb{N}}$ which is a small v-sheaf. We can prove inductively that $B^+_{dR}(\mathcal{O}^\sharp)/\xi^n$ is a small v-sheaf. Indeed, it sits in the exact sequence of presheaves:

$$0 \to B^+_{dR}(\mathcal{O}^\sharp)/\xi^{n-1} \xrightarrow{\cdot\xi} B^+_{dR}(\mathcal{O}^\sharp)/\xi^n \to \mathcal{O}^\sharp \to 0$$

By induction the leftmost term is a small v-sheaf and we already know that the rightmost term is a small v-sheaf. A diagram chase gives that the middle one is also a small v-sheaf. Since $B^+_{dR}(\mathcal{O}^\sharp) = \varprojlim_n B^+_{dR}(\mathcal{O}^\sharp)/\xi^n$ this other one is also a small v-sheaf.

Now $W(\mathcal{O}) = \varinjlim(W^+(\mathcal{O}) \xrightarrow{\xi} W^+(\mathcal{O}) \xrightarrow{\xi} \dots)$ and $B_{dR}(\mathcal{O}^\sharp) = \varinjlim(B^+_{dR}(\mathcal{O}^\sharp) \xrightarrow{\xi} B^+_{dR}(\mathcal{O}^\sharp) \xrightarrow{\xi} \dots)$. Since these are filtered colimit of sheaves they define small v-sheaves as well. □

Notice that $W^+(\mathcal{O})$ and $B^+_{dR}(\mathcal{O}^\sharp)$ come equipped with reduction maps

$$red : W^+(\mathcal{O}) \to W^+(\mathcal{O})/\xi = \mathcal{O}^{\sharp,+}$$

and

$$red : B^+_{dR}(\mathcal{O}^\sharp) \to B^+_{dR}(\mathcal{O}^\sharp)/\xi = \mathcal{O}^\sharp.$$

Definition 2.2.14. *Let* H *be a finite type affine scheme over* $\mathrm{Spec}(W(k)[t, t^{-1}])$*, and let* $(\mathcal{H}, \rho)$ *be a finite type affine scheme over* $\mathrm{Spec}(W(k)[t])$ *together with an isomorphism* $\rho : \mathcal{H} \times_{\mathbb{A}^1_{W(k)}} \mathbb{G}_m \to H$*. To this setup we associate the following presheaves over* $O_C^\lozenge$*:*

1. $W^+\mathcal{H}$ *assigns to* $\mathrm{Spa}(R, R^+) \to O_C^\lozenge$ *the set of sections* $\mathrm{Spec}(W(R^+)) \to \mathcal{H} \times_{\mathbb{A}^1_{W(k)}} \mathrm{Spec}(W(R^+))$*.*

2. WH *assigns to* $\mathrm{Spa}(R, R^+) \to O_C^\lozenge$ *the set of sections* $\mathrm{Spec}(W(R^+)[\frac{1}{\xi}]) \to H \times_{\mathbb{G}_{m,W(k)}} \mathrm{Spec}(W(R^+)[\frac{1}{\xi}])$*.*

3. $L^+\mathcal{H}$ *assigns to* $\mathrm{Spa}(R,R^+) \to O_C^\lozenge$ *the set of sections* $\mathrm{Spec}(B^+_{dR}(R^\sharp)) \to \mathcal{H} \times_{\mathbb{A}^1_{W(k)}} \mathrm{Spec}(B^+_{dR}(R^\sharp))$.

4. LH *assigns to* $\mathrm{Spa}(R,R^+) \to O_C^\lozenge$ *the set of sections* $\mathrm{Spec}(B_{dR}(R^\sharp)) \to H \times_{\mathbb{G}_{m,W(k)}} \mathrm{Spec}(B_{dR}(R^\sharp))$.

where the base change in all cases is given by the usual map on $W(k)$ *deduced from the composition* $k \to O_C \to R^+$ *and given by* $t \mapsto \xi$.

Proposition 2.2.15. *With the notation as above* $W^+\mathcal{H}$, WH, $L^+\mathcal{H}$ *and* LH *are small* v*-sheaves.*

Proof. Let $\mathcal{R} \in \{W^+(\mathcal{O}), W(\mathcal{O}), B^+_{dR}(\mathcal{O}^\sharp), B_{dR}(\mathcal{O}^\sharp)\}$ denote one of the sheaves of rings of definition 2.2.12. Suppose that $\mathcal{H} = \mathrm{Spec}(W(k)[t][x_1, \dots x_n]/(f_1(t,\overline{x}), \dots, f_m(t,\overline{x}))$. Notice there is a map of v-sheaves $O_C^\lozenge \xrightarrow{0} \mathcal{R}$ corresponding to the constant 0 section. Consider the following basechange diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & O_C^\lozenge \arrow[d, "0"] \\ \mathcal{R}^n \arrow[r, "F"] & \mathcal{R}^m \end{tikzcd}$$

Where $F(\overline{r}) = (f_1(\xi, \overline{r}), \dots, f_m(\xi, \overline{r}))$. Whenever $\mathcal{R}$ is $W^+(\mathcal{O})$, $W(\mathcal{O})$, $B^+_{dR}(\mathcal{O}^\sharp)$, or $B_{dR}(\mathcal{O}^\sharp)$ then X is isomorphic as presheaves to $W^+\mathcal{H}$, WH, $L^+\mathcal{H}$, or LH respectively. From this diagram, it is clear that X is a small v-sheaf. $\square$

In our setup, ρ will induce maps of v-sheaves $L^+\mathcal{H} \xrightarrow{\rho} LH$ and $W^+\mathcal{H} \xrightarrow{\rho} WH$. We get the following diagrams of inclusions:

$$\begin{tikzcd} & W(\mathcal{O}) \arrow[dr, hook] & \\ W^+(\mathcal{O}) \arrow[ur, hook] \arrow[dr, hook] & & B_{dR}(\mathcal{O}^\sharp) \\ & B^+_{dR}(\mathcal{O}^\sharp) \arrow[ur, hook] & \end{tikzcd} \qquad \begin{tikzcd} & WH \arrow[dr, hook] & \\ W^+\mathcal{H} \arrow[ur, hook, "\rho"] \arrow[dr, hook] & & LH \\ & L^+\mathcal{H} \arrow[ur, hook, "\rho"'] & \end{tikzcd}$$

Moreover, if we let $\overline{\mathcal{H}}$ denote the basechange $\mathcal{H} \times_{\mathbb{A}^1_{W(k)}} \mathrm{Spec}(W(k))$ at $t = 0$ we get reduction morphisms $W^+\mathcal{H} \to \overline{\mathcal{H}}(\mathcal{O}^{\sharp,+})_{O_C^\lozenge}$ and $L^+\mathcal{H} \to \overline{\mathcal{H}}(\mathcal{O}^\sharp)_{O_C^\lozenge}$.

Proposition 2.2.16. *If* $\mathcal{H}$ *is smooth over* $\mathrm{Spec}(W(k)[t])$ *then the reduction maps* $W^+\mathcal{H} \to \overline{\mathcal{H}}(\mathcal{O}^{\sharp,+})_{O_C^\lozenge}$ *and* $L^+\mathcal{H} \to \overline{\mathcal{H}}(\mathcal{O}^\sharp)_{O_C^\lozenge}$ *are surjective maps of* v*-sheaves.*

Proof. We claim that the map is surjective even at the level of presheaves. The (R, R^+)-valued points of $\overline{\mathcal{H}}(\mathcal{O}^\sharp)$ and $\overline{\mathcal{H}}(\mathcal{O}^{\sharp,+})$ can be seen as maps

$$\mathrm{Spec}(R^\sharp) \to \mathcal{H}_{B_{dR}(R^\sharp)}$$

and

$$\mathrm{Spec}(R^{\sharp,+}) \to \mathcal{H}_{W(R^+)}$$

whose composition with the projections to $\mathrm{Spec}(B_{dR}(R^\sharp))$ and $\mathrm{Spec}(W(R^+))$ are the usual closed embeddings. By smoothness of $\mathcal{H}$, for any $n \in \mathbb{N}$ the maps can be lifted to maps $\mathrm{Spec}(B_{dR}(R^\sharp)/\xi^n) \to \mathcal{H}_{B_{dR}(R^\sharp)}$ and $\mathrm{Spec}(W(R^+)/\xi^n) \to \mathcal{H}_{W(R^+)}$ respectively. Since $\mathcal{H}$ is an affine scheme and since both $B_{dR}(R^\sharp)$ and $W(R^+)$ are (ξ)-adically complete we may pass to the inverse limit by choosing compatible lifts. $\square$

Definition 2.2.17. *1. With the setup as above, consider the ring k_C with the discrete topology, we let $\mathcal{W}^+_{\mathrm{red}}\mathcal{H} \in \widetilde{\mathrm{SchPerf}}_{k_C}$ be the scheme-theoretic v-sheaf that assigns to $\mathrm{Spec}(R) \in \mathrm{PCAlg}^{op}_{k_C}$ sections*

$$\mathrm{Spec}(W(R)) \to \mathcal{H} \times_{\mathbb{A}^1_{W(k)}} \mathrm{Spec}(W(R))$$

where the basechange is given by $t \mapsto p$.

2. We let $\mathcal{W}^+_{\mathrm{red}}(\mathcal{O}) : \mathrm{PCAlg}^{op}_{k_C} \to \mathrm{Sets}$ denote sheaf that sends $\mathrm{Spec}(R)$ to $W(R)$. This sheaf can also be expressed as $\mathcal{W}^+_{\mathrm{red}}\mathbb{A}^1_{W(k)[t]}$.

Remark 2.2.18. *The scheme-theoretic v-sheaves $\mathcal{W}^+_{\mathrm{red}}\mathcal{H}$ of definition 2.2.17 are the v-sheaves that Zhu calls p-adic jet spaces in [59] 1.1.1. These sheaves are represented by perfect affine schemes.*

Proposition 2.2.19. *With the notation as above the v-sheaf $W^+\mathcal{H}$ is a p-adic kimberlite and $(W^+\mathcal{H})^{\mathrm{red}} = (\mathcal{W}^+_{\mathrm{red}}\mathcal{H})$.*

Proof. Observe that $W^+(\mathcal{O})$ is represented by $\mathrm{Spd}(O_C\langle T_n\rangle_{n\in\mathbb{N}}, O_C\langle T_n\rangle_{n\in\mathbb{N}})$, by proposition 1.4.23 and proposition 1.3.25 $W^+(\mathcal{O})$ is a p-adic kimberlite. Moreover, by proposition 1.3.20 $W^+(\mathcal{O})^{\mathrm{red}}$ is represented by $\mathrm{Spec}(k_C[T_n]_{n\in\mathbb{N}})$ which is $\mathcal{W}^+_{\mathrm{red}}(\mathcal{O})$. Lets move on to the general case, recall from the proof of proposition 2.2.15 that if $\mathcal{H} = \mathrm{Spec}(A)$ is presented as $A = W(k)[t][\overline{x}]/I$ with $I = (f_1(t, \overline{x}), \dots, f_m(t, \overline{x}))$. Then $W^+\mathcal{H}$ fits in the commutative diagram with Cartesian square:

$$\begin{tikzcd}
W^+\mathcal{H} \arrow[r] \arrow[d] & O_C^\diamondsuit \arrow[d, "0"] \arrow[dr, "id"] & \\
W^+(\mathcal{O})^n \arrow[r, "F"] & W^+(\mathcal{O})^m \arrow[r] & O_C^\diamondsuit
\end{tikzcd}$$

We claim that all of these maps are formally adic, and in particular $W^+\mathcal{H}$ is formally p-adic. This follows from the fact that formal adicness is stable under basechange, that it has the 2-out-of-3 property and that, as we proved above, $W^+(\mathcal{O}) \to O_C^\lozenge$ is formally adic. Since $W^+(\mathcal{O})$ is separated over $O_C^\lozenge$ the section $O_C^\lozenge \xrightarrow{0} W^+(\mathcal{O})^m$ is a formally adic closed immersion. We can conclude that $W^+\mathcal{H}$ is a p-adic kimberlite by using lemma 1.4.30. Finally, since we can basechange by $\mathrm{Spec}(k_C)^\lozenge \to \mathrm{Spd}(O_C, O_C)$ to compute reductions we get the following Cartesian diagram:

$$\begin{tikzcd}
W^+\mathcal{H}^{\mathrm{red}} \arrow[r] \arrow[d] & \mathrm{Spec}(k_C) \arrow[d, "0"] \\
\mathcal{W}^+_{\mathrm{red}}(\mathcal{O})^n \arrow[r, "F"] & \mathcal{W}^+_{\mathrm{red}}(\mathcal{O})^m
\end{tikzcd}$$

which gives the isomorphism $W^+\mathcal{H}^{\mathrm{red}} = \mathcal{W}^+_{\mathrm{red}}\mathcal{H}$.

□

2.2.3 Demazure kimberlites

In this subsection we use twisted loop sheaves to construct a family of kimberlites that will allow us to understand how the specialization map for the p-adic Beilinson-Drinfeld Grassmanians behave. We change the setup a little bit and fix some notation first:

1. Let H be a split reductive group over $W(k)$, let $T \subseteq B \subseteq H$ a choice of maximal split torus and a Borel respectively.
2. Let $(X^*, \Phi, X_*, \Phi^\vee)$ be the root datum associated to (H, T).
3. We let $\langle \cdot, \cdot \rangle : X^* \times X_* \to \mathbb{Z}$ denote perfect pairing between roots and coroots.
4. Let Φ^+ be the set of positive roots associated to B.
5. Let N be the normalizer of T in H.
6. Let $W = N/T$ be the Weyl group of H.
7. We let $\mathcal{A} = \mathcal{A}(H, T)$ denote $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$.
8. We let $\Psi = \{\alpha + n \mid \alpha \in \Phi,\, n \in \mathbb{Z}\}$ denote the set of affine functionals on $\mathcal{A}$ coming from the natural perfect pairing $\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$. We call them affine roots.
9. Given a point $q \in \mathcal{A}$ we let $\Phi_q = \{\alpha \in \Phi \mid \alpha(q) \in \mathbb{Z}\}$ this is clearly a closed sub-root system. We let M_q be the generalized Levi subgroup of H containing T with root datum given by $(X^*, \Phi_q, X_*, \Phi_q^\vee)$.
10. Ψ defines a hyperplane structure on $\mathcal{A}$, and for any point $q \in \mathcal{A}$ we can associate a polysimplicial closed region of $\mathcal{A}$ that we will denote by F_q. In case H is semisimple this region is bounded and forms a polytope. We let o denote the vertex associated to the origin in $\mathcal{A}$ and $\mathcal{C}$ the unique alcove containing o and contained in the Bruhat chamber associated to B.

11. We denote by $\mathbb{S}$ the set of reflections along the walls of $\mathcal{C}$, we let W^{aff} the affine Weyl group generated by $\mathbb{S}$. Given any facet $\mathcal{F} \subseteq \mathcal{C}$ we let $\mathbb{S}_{\mathcal{F}}$ be the subset of elements of $\mathbb{S}$ fixing $\mathcal{F}$ and we let $W_{\mathcal{F}}$ be the subgroup of W^{aff} generated by $\mathbb{S}_{\mathcal{F}}$.

12. We let $\tilde{W}$ denote the Iwahori-Weyl group of H. Recall that $W^{aff} \subseteq \tilde{W}$ and that if we let $\Omega_H = \pi_1(H^{der})$ we have a decomposition $\tilde{W} = W^{aff} \rtimes \Omega_H$.

Fix a point $q \in \mathcal{A}$. In ([42] §3) Pappas and Zhu use Bruhat-Tits theory and dilatation techniques to construct smooth affine algebraic groups $\mathcal{H}_q$ over $\mathrm{Spec}(W(k)[t])$ together with an isomorphism ρ from $\mathcal{H}_q \times_{W(k)[t]} \mathrm{Spec}(W(k)[t, t^{-1}])$ to $H \times_{W(k)} \mathrm{Spec}(W(k)[t, t^{-1}])$ with the following list of properties:

a) For any discrete valuation ring V and a map $W(k)[t] \to V$ given by $t \mapsto \pi$ with $\pi \in V$ a uniformizer, the basechange $\mathcal{H}_q \times_{\mathbb{A}^1_{W(k)}} \mathrm{Spec}(V)$ is the parahoric group scheme associated to $q \in \mathcal{A}(H, V[\frac{1}{\pi}])$ by Bruhat-Tits theory under the identification $\mathcal{A}(H, V[\frac{1}{\pi}]) = X^*(T_{V[\frac{1}{\pi}]}) \otimes \mathbb{R} = X^*(T) \otimes \mathbb{R}$.

b) For any root $\alpha \in \Phi$ there are smooth connected closed subgroups $\mathcal{U}^q_\alpha \subseteq \mathcal{H}_q$ (respectively $\mathcal{T} \subseteq \mathcal{H}_q$) extending the usual root subgroup $U_\alpha \subseteq H$ (respectively extending the torus $T \subseteq H$). Over $W(k)[t]$, the groups $\mathcal{U}^q_\alpha$ are isomorphic to $\mathbb{G}_a$ and $\mathcal{T}$ is isomorphic to $\mathbb{G}_m^n$ for some n.

c) There is an open cell decomposition:

$$\mathcal{V}_q := \prod_{\alpha \in \Phi^-} \mathcal{U}^q_\alpha \times \mathcal{T} \times \prod_{\alpha \in \Phi^+} \mathcal{U}^q_\alpha \to \mathcal{H}_q.$$

This map forms an open embedding onto a fiberwise Zariski-dense neighborhood of the identity section.

d) The group multiplication map $\mathcal{V}_q \times \mathcal{V}_q \xrightarrow{\mu} \mathcal{H}_q$ is smooth and surjective.

e) The basechange $\overline{\mathcal{H}}_q := \mathcal{H}_q \times_{\mathbb{A}^1_{W(k)}} \mathrm{Spec}(W(k))$ along $t = 0$ supports a split reductive quotient $\overline{\mathcal{H}}^{Red}_q$ over $W(k)$ with root datum canonically identified with $(X^*, \Phi_q, X_*, \Phi_q^\vee)$. In particular we can identify M_q with $\overline{\mathcal{H}}^{Red}_q$.

f) If $\alpha \in \Phi_q$ the composition $\overline{\mathcal{U}}^q_\alpha \to \overline{\mathcal{H}}^{Red}_q$ at $t = 0$ defines an isomorphism onto the root group of $\overline{\mathcal{H}}^{Red}_q$ corresponding to α. On the other hand, if $\alpha \in \Phi \setminus \Phi_q$ then the composition $\overline{\mathcal{U}}^q_\alpha \to \overline{\mathcal{H}}^{Red}_q$ factors through the identity section.

g) We have a commutative diagram of open cell decompostion:

$$\begin{tikzcd}
\prod_{\alpha\in\Phi\setminus\Phi_q}\overline{\mathcal{U}}_\alpha \arrow[r] \arrow[d, "\cong"] & \prod_{\alpha\in\Phi^-}\overline{\mathcal{U}}_\alpha\times\overline{\mathcal{T}}\times\prod_{\alpha\in\Phi^+}\overline{\mathcal{U}}_\alpha \arrow[r, "\pi"] \arrow[d, "\mu"] & \prod_{\alpha\in\Phi_q^-}\overline{\mathcal{U}}_\alpha\times\overline{\mathcal{T}}\times\prod_{\alpha\in\Phi_q^+}\overline{\mathcal{U}}_\alpha \arrow[d, "\mu"] \\
Ker(m) \arrow[r] & \overline{\mathcal{H}}_q \arrow[r, "m"] & \overline{\mathcal{H}}_q^{Red}
\end{tikzcd}$$

where π is the projection map and the left vertical arrow is an isomorphism.

If we are given two points $q_1, q_2 \in \mathcal{A}$ such that $F_{q_1} \subseteq F_{q_2}$ we also get a map of algebraic groups $f : \mathcal{H}_{q_2} \to \mathcal{H}_{q_1}$. This map has the following properties:

a) $\rho_1 \circ f = \rho_2$ over $W(k)[t, t^{-1}]$.

b) The composition $\overline{\mathcal{H}}_{q_2} \to \overline{\mathcal{H}}_{q_1}^{Red}$ surjects onto the parabolic subgroup of $\overline{\mathcal{H}}_{q_1}^{Red}$ associated to the closed sub-root system given by $\Phi q_1.q_2 := \{\alpha \in \Phi_{q_1} \mid \lfloor\alpha(q_2)\rfloor = \alpha(q_1)\}$. Moreover, the kernel of this map is fiberwise a vector group.

We are now prepared to define "parahoric" versions of the positive loop groups which we will use to define Demazure kimberlites.

Definition 2.2.20. *1. We define the loop group LH to be as in definition 2.2.14 when we consider H as a scheme over $W(k)[t, t^{-1}]$ by taking the appropriate basechange.*

2. Given a point $q \in \mathcal{A}$ define the parahoric loop group to be $L^+\mathcal{H}_q$ as in definition 2.2.14.

3. Associated to the same point we also define the formal parahoric loop group to be $W^+\mathcal{H}_q$.

Notice that we have injective maps of v-sheaves $W^+\mathcal{H}_q \subseteq L^+\mathcal{H}_q \overset{\rho}{\subseteq} LH$.

Proposition 2.2.21. *With the notation as above, for any point $q \in \mathcal{A}$ we have surjective morphisms of v-sheaves in groups:*

$$L^+\mathcal{H}_q \to \overline{\mathcal{H}}_q^{Red}(\mathcal{O}_X^\sharp) = M_q(\mathcal{O}_X^\sharp)$$

$$W^+\mathcal{H}_q \to \overline{\mathcal{H}}_q^{Red}(\mathcal{O}_X^{\sharp,+}) = M_q(\mathcal{O}_X^{\sharp,+})$$

Proof. This is a direct consequence of proposition 2.2.16 and proposition 2.2.10 since the map $\overline{\mathcal{H}} \to \overline{\mathcal{H}}^{Red}$ is smooth surjective and consequently universally subtrusive. $\square$

We let $L^u\mathcal{H}_q$ and $W^u\mathcal{H}_q$ denote respectively the kernels of the morphisms of proposition 2.2.21 above.

Proposition 2.2.22. *If $q_1, q_2 \in \mathcal{A}$ are such that $F_{q_1} \subseteq F_{q_2}$, then we get inclusions of v-sheaves in groups:*

$$L^u\mathcal{H}_{q_1} \subseteq L^u\mathcal{H}_{q_2} \subseteq L^+\mathcal{H}_{q_2} \subseteq L^+\mathcal{H}_{q_1} \subseteq LH$$

$$W^u\mathcal{H}_{q_1} \subseteq W^u\mathcal{H}_{q_2} \subseteq W^+\mathcal{H}_{q_2} \subseteq W^+\mathcal{H}_{q_1} \subseteq WH$$

Moreover, the map from $L^+\mathcal{H}_{q_2}$ to $M_{q_1}(\mathcal{O}^\sharp)$ surjects onto $P_{\Phi_{q_1,q_2}}(\mathcal{O}^\sharp) \subseteq M_{q_1}(\mathcal{O}^\sharp)$. Analogously, $W^+\mathcal{H}_{q_2}$ surjects onto $P_{\Phi_{q_1,q_2}}(\mathcal{O}^{\sharp,+}) \subseteq M_{q_1}(\mathcal{O}^{\sharp,+})$.

Proof. We will deal only with the case of parahoric loop groups since the other case is completely analogous. Recall that we have a morphism of algebraic groups over $W(k)[t]$, $f : \mathcal{H}_{q_2} \to \mathcal{H}_{q_1}$, such that $\rho_1 \circ f = \rho_2$. Functoriality of L^+, gives us maps $L^+\mathcal{H}_{q_2} \to L^+\mathcal{H}_{q_1} \to LH$, since $L^+\mathcal{H}_{q_2} \to LH$ is an injection, then $L^+\mathcal{H}_{q_2} \to L^+\mathcal{H}_{q_1}$ is also injective.

Now, since the map of affine schemes $\overline{\mathcal{H}}_{q_2} \to P_{\Phi_{q_1,q_2}}$ is faithfully flat of finite presentation it is universally subtrusive. This implies, by proposition 2.2.16 and proposition 2.2.10, that the composition of $L^+\mathcal{H}_{q_2} \to \overline{\mathcal{H}}_{q_2}(\mathcal{O}^\sharp)$ with $\overline{\mathcal{H}}_{q_2}(\mathcal{O}^\sharp) \to P_{\Phi_{q_1,q_2}}(\mathcal{O}^\sharp)$ is surjective.

Finally, we claim that any map $g : \mathrm{Spec}(B^+_{dR}(R^\sharp)) \to \mathcal{H}_{q_1,B^+_{dR}(R^\sharp)}$ whose reduction $\mathrm{Spec}(R^\sharp) \to M_q$ factors through the identity section lifts to a map $\mathrm{Spec}(B^+_{dR}(R^\sharp)) \to \mathcal{H}_{q_2,B^+_{dR}(R^\sharp)}$. Indeed, observe that $\mathrm{Spec}(R^\sharp) \to \overline{\mathcal{H}}_{q_1}$ factors through the open cell $\overline{\mathcal{V}}_{q_1}$, which implies that g is of the form

$$g = (\prod_{\alpha\in\Phi^-_{q_1}} u_\alpha(g)) \cdot t(g) \cdot (\prod_{\alpha\in\Phi^+_{q_1}} u_\alpha(g))$$

with $t(g)$ and $\{u_\alpha(g)\}_{\alpha\in\Phi\setminus\Phi_{q_1}}$ reducing to the identity.

We can verify directly from the construction of the map $\mathcal{H}_{q_2} \to \mathcal{H}_{q_1}$ that each of this elements lifts uniquely to an element in $\mathcal{V}_{q_2}$. Indeed, on the torus $\mathcal{T}$ and on $\mathcal{U}_\alpha$ with $\alpha \notin \Phi_{q_1}$ f induces an isomorphism because $F_{q_1} \subseteq F_{q_2}$. For $\alpha \in \Phi_{q_1} \setminus \Phi_{q_1,q_2}$ we may, after making some choices, write $\mathcal{U}^{q_1}_\alpha$ as $\mathrm{Spec}(W(k)[t,u])$ and $\mathcal{U}^{q_2}_\alpha$ as $\mathrm{Spec}(W(k)[t,\frac{u}{t}])$. In this case f restricted to $\mathcal{U}^{q_2}_\alpha$ is given by the natural inclusion of rings. The map of rings $u_\alpha(g)^* : W(k)[t,u] \to B^+_{dR}(R^\sharp)$ with $t \mapsto \xi$ extends to a map $W(k)[t,\frac{u}{t}] \to B^+_{dR}(R^\sharp)$ whenever ξ divides the image of u, but this happens whenever $u_\alpha(g)$ reduces to identity. □

Proposition 2.2.23. *Let $q_1, q_2 \in \mathcal{A}$ such that $F_{q_1} \subseteq F_{q_2}$, then we have isomorphisms of quotient v-sheaves $L^+\mathcal{H}_{q_1}/L^+\mathcal{H}_{q_2} = W^+\mathcal{H}_{q_1}/W^+\mathcal{H}_{q_2}$. Moreover, we can identify both of these quotients with $(Fl_{q_1,q_2,O_C})^\diamondsuit$, where Fl_{q_1,q_2} denotes the flag variety $M_{q_1}/P_{\Phi_{q_1,q_2}}$ when thought of as a p-adic formal scheme.*

Proof. We have sequence of equalities:

$$\begin{aligned} L^+\mathcal{H}_{p_1}/L^+\mathcal{H}_{p_2} &= (L^+\mathcal{H}_{p_1}/L^u\mathcal{H}_{p_1})/(L^+\mathcal{H}_{p_2}/L^u\mathcal{H}_{p_1}) \\ &= M_{p_1}(\mathcal{O}^\sharp)/P_{\Phi_{p_1,p_2}}(\mathcal{O}^\sharp) \\ &= Fl_{p_1,p_2}(\mathcal{O}^\sharp) \\ &= Fl^\diamondsuit_{p_1,p_2} \end{aligned}$$

The last equality follows from proposition 2.2.9.

Analogously:

$$\begin{aligned} W^+\mathcal{H}_{p_1}/W^+\mathcal{H}_{p_2} &= (W^+\mathcal{H}_{p_1}/W^u\mathcal{H}_{p_1})/(W^+\mathcal{H}_{p_2}/W^u\mathcal{H}_{p_1}) \\ &= M_{p_1}(\mathcal{O}^{\sharp,+})/P_{\Phi_{p_1,p_2}}(\mathcal{O}^{\sharp,+}) \\ &= Fl^{\Diamond}_{p_1,p_2} \end{aligned}$$

□

Lemma 2.2.24. *1. Let $\mathcal{F}$ be a locally spatial diamond with a map $\mathcal{F} \to O_C^{\Diamond}$ and fix two points $q_1, q_2 \in \mathcal{A}$ with $F_{q_1} \subseteq F_{q_2}$. The natural map $L^+\mathcal{H}_{q_1} \times_{O_C^{\Diamond}} \mathcal{F} \to Fl^{\Diamond}_{q_1,q_2} \times_{O_C^{\Diamond}} \mathcal{F}$ admits pro-étale locally a section.*

2. If $\mathrm{Spa}(R, R^+)$ is affinoid perfectoid and we are given a map $\mathrm{Spa}(R, R^+) \to Fl^{\Diamond}_{q_1,q_2}$ then the pullback $L^+\mathcal{H}_{q_1} \times_{Fl^{\Diamond}_{q_1,q_2}} \mathrm{Spa}(R, R^+) \to \mathrm{Spa}(R, R^+)$ admits a section locally on the analytic topology of $\mathrm{Spa}(R, R^+)$

Proof. We may reduce the first claim to the second one by [51] 11.24. Indeed, by proposition 2.2.23 the map in question forms a $L^+\mathcal{H}_{q_2}$-torsor so it is enough to prove it is pro-étale locally the trivial torsor. Since the map $Fl^{\Diamond}_{q_1,q_2} \to O_C^{\Diamond}$ is representable in spatial diamonds we can find a pro-étale cover $\mathrm{Spa}(R, R^+) \to Fl^{\Diamond}_{q_1,q_2} \times_{O_C^{\Diamond}} \mathcal{F}$ with $\mathrm{Spa}(R, R^+)$ affinoid perfectoid.

Let us prove the second claim. The obstruction to the triviality of the $L^+\mathcal{H}_{q_2}$-torsor over $\mathrm{Spa}(R, R^+)$ is an element $\mathfrak{obs}$ in $H^1_v(\mathrm{Spa}(R, R^+), L^+\mathcal{H}_{q_2})$. We prove that this obstruction vanishes after a localization in the analytic topology, recall the following two sequences of maps:

$$(L^+\mathcal{H}_{q_1}) \to M_{q_1}(\mathcal{O}_X^{\sharp}) \to Fl^{\Diamond}_{q_1,q_2}$$

$$e \to L^u\mathcal{H}_{q_1} \to L^+\mathcal{H}_{q_2} \to P_{\Phi_{q_1,q_2}}(\mathcal{O}_X^{\sharp}) \to e$$

The map $M_{q_1}(\mathcal{O}_X^{\sharp}) \to Fl^{\Diamond}_{q_1,q_2}$ is a $P_{\Phi_{q_1,q_2}}(\mathcal{O}_X^{\sharp})$-torsor with obstruction to triviality lying in

$$H^1_v(Fl^{\Diamond}_{q_1,q_2}, P_{\Phi_{q_1,q_2}}(\mathcal{O}_X^{\sharp})).$$

Since the map of schemes $M_{q_1} \to Fl_{q_1,q_2}$ admits Zariski locally a section we may replace $\mathrm{Spa}(R, R^+)$ by an analytic cover for which $\mathfrak{obs}$ in $H^1_v(\mathrm{Spa}(R, R^+), P_{\Phi_{q_1,q_2}}(\mathcal{O}_X^{\sharp}))$ is trivial. Observe that $\mathfrak{obs}$ comes from an element of $H^1_v(\mathrm{Spa}(R, R^+), L^u\mathcal{H}_{q_1})$, we prove that in this case it is already trivial. Consider the exact sequence

$$e \to Ker\left(L^+\mathcal{H}_{q_1} \to \overline{\mathcal{H}}_{q_1}(\mathcal{O}^{\sharp})\right) \to L^u\mathcal{H}_{q_1} \to Ker\left(\overline{\mathcal{H}}_{q_1}(\mathcal{O}^{\sharp}) \to \overline{\mathcal{H}}^{Red}_{q_1}(\mathcal{O}^{\sharp})\right) \to e,$$

we prove that after applying $H^1(\mathrm{Spa}(R, R^+), -)$ to the two groups in the extremes we obtain the trivial pointed set.

For the first group we construct a family $\{L^{u,n}\}_{n=1}^{\infty}$ of groups filtering

$$L^{u,1} := Ker\left(L^+\mathcal{H}_{q_1} \to \overline{\mathcal{H}}_{q_1}(\mathcal{O}^\sharp)\right).$$

We define them as:

$$L^{u,n}(R,R^+) := Ker\left(\mathcal{H}_{q_1,B^+_{dR}(R^\sharp)}(B^+_{dR}(R^\sharp)) \to \mathcal{H}_{q_1,B^+_{dR}(R^\sharp)}(B^+_{dR}(R^\sharp)/\xi^n)\right)$$

Successive quotients $L^{u,n}/L^{u,n+1}$ get identified with the sheaves assigning to $\mathrm{Spa}(R,R^+)$ the groups:

$$Ker\left(\mathcal{H}_{q_1,B^+_{dR}(R^\sharp)}(B^+_{dR}(R^\sharp)/\xi^{n+1}) \to \mathcal{H}_{q_1,B^+_{dR}(R^\sharp)}(B^+_{dR}(R^\sharp)/\xi^n)\right)$$

Since $\mathrm{Spec}(B^+_{dR}(R^\sharp)/\xi^n) \to \mathrm{Spec}(B^+_{dR}(R^\sharp)/\xi^{n+1})$ is a first order nilpotent thickening, deformation theory gives:

$$L^{u,n}/L^{u,n+1} = Hom(e^*\Omega^1_{\mathcal{H}_{q_1}} \otimes_{W(k)[t]} B^+_{dR}(R^\sharp), (\xi^n \cdot B^+_{dR}(R^\sharp)/\xi^{n+1})) = Hom(e^*\Omega^1_{\mathcal{H}_{q_1}} \otimes R^\sharp, R^\sharp)$$

Since $\mathcal{H}_{q_1}$ has an open cell decomposition we can see explicitly that $e^*\Omega^1_{\mathcal{H}_{q_1}/W(k)[t]}$ is a finite free module over $W(k)[t]$ (a priori it is only projective), and after fixing a basis we get an identification $L^{u,n}/L^{u,n+1} = (\mathcal{O}^\sharp)^n$. By ([51] 8.8) the cohomology group $H^1_v(\mathrm{Spa}(R,R^+),\mathcal{O}^\sharp) = 0$. Let $I^{u,n}$ be the image of $H^1_v(\mathrm{Spa}(R,R^+),L^{u,n})$ in $H^1_v\left(\mathrm{Spa}(R,R^+),L^{u,1}\right)$. The argument above shows that $\mathfrak{obs} \in \bigcap_{n\in\mathbb{N}} I^{u,n}$. On the other hand,

$$L^{u,1} = \varprojlim L^{u,1}/L^{u,n}$$

with transition maps that are surjective at the level of presheaves. One can use Čech cohomology to prove that $\bigcap_{n\in\mathbb{N}} I^{u,n} = \{e\}$. So far we have proved $H^1_v\left(\mathrm{Spa}(R,R^+),L^{u,1}\right) = \{e\}$, but

$$H^1_v\left(\mathrm{Spa}(R,R^+), Ker(\overline{\mathcal{H}}_{q_1}(\mathcal{O}^\sharp) \to \overline{\mathcal{H}}^{Red}_{q_1}(\mathcal{O}^\sharp))\right)$$

is also trivial since $Ker(\overline{\mathcal{H}}_{q_1} \to \overline{\mathcal{H}}^{Red}_{q_1})$ is a vector group over $W(k)$ and we may use [51] 8.8 again. This proves that $H^1_v(\mathrm{Spa}(R,R^+),L^u\mathcal{H}_{q_1}) = \{e\}$ and finishes the proof. $\square$

Definition 2.2.25. *Let $\sigma_r := \{r_i\}_{1\leq i\leq n}$ and $\sigma_q := \{q_i\}_{1\leq i\leq n}$ denote a pair of sequences of points in $\mathcal{A}$ such that $F_{r_i}, F_{r_{i+1}} \subseteq F_{q_i}$, and let σ denote the tuple (σ_r,σ_q). To each σ of this form we associate a v-sheaf that we call the Demazure kimberlite of σ. We define them as the contracted group product:*

$$D(\sigma) = L^+\mathcal{H}_{r_1} \overset{L^+\mathcal{H}_{q_1}}{\times_{O_C^\diamond}} L^+\mathcal{H}_{r_2} \overset{L^+\mathcal{H}_{q_2}}{\times_{O_C^\diamond}} \dots \overset{L^+\mathcal{H}_{q_{n-1}}}{\times_{O_C^\diamond}} L^+\mathcal{H}_{r_n}/L^+\mathcal{H}_{q_n}$$

In what follows we will prove that for any σ as above the $D(\sigma)$ are rich p-adic kimberlites that are proper and smooth over $O_C^\diamond$.

Proposition 2.2.26. *The map of v-sheaves $D(\sigma) \to O_C^\diamondsuit$ is representable in spatial diamonds, proper and ℓ-cohomologically smooth for any $\ell \neq p$.*

Proof. Let σ be as above and let $\sigma' = (\{r_i\}_{1\leq i\leq n-1}, \{q_i\}_{1\leq i\leq n-1})$ be the subsequence of the first $n-1$ points of σ. We have a projection morphism of v-sheaves $f : D(\sigma) \to D(\sigma')$ given by forgetting the last entry corresponding to r_n. It is enough to inductively show that this map satisfies all of the properties in the hypothesis. Since the definition of $D(\sigma)(R, R^+)$ is independent of R^+ to prove the map is proper it is enough to prove it is quasi-compact and separated over $D(\sigma')$. By ([51] 23.15, 10.11, 13.4) separatedness, quasi-compactness and almost all of the requirements that a map needs to satisfy to be ℓ-cohomologically smooth ([51] 23.8) can be checked v-locally. The following diagram is Cartesian with surjective horizontal arrows:

$$\begin{tikzcd} L^+\mathcal{H}_{r_1}\times_{O_C^\diamondsuit}\cdots\times_{O_C^\diamondsuit} L^+\mathcal{H}_{r_{n-1}}\times_{O_C^\diamondsuit}(L^+\mathcal{H}_{r_n}/L^+\mathcal{H}_{q_n}) \arrow[r] \arrow[d] & D(\sigma) \arrow[d] \\ L^+\mathcal{H}_{r_1}\times_{O_C^\diamondsuit}\cdots\times_{O_C^\diamondsuit} L^+\mathcal{H}_{r_{n-1}} \arrow[r] & D(\sigma') \end{tikzcd}$$

By proposition 2.2.23 we have that $L^+\mathcal{H}_{r_n}/L^+\mathcal{H}_{q_n} = Fl^\diamondsuit_{r_n,q_n,O_C}$ which is proper, representable in spatial diamonds and ℓ-cohomologically smooth over $O_C^\diamondsuit$ for any $\ell \neq p$. This proves the map in the left column of the diagram is representable in spatial diamonds, ℓ-cohomologically smooth and proper. It also proves that the map in the right column of the diagram satisfy the properties that can be checked v-locally.

We now verify the finer properties that cannot be checked v-locally. Namely, we need to check that $f : D(\sigma) \to D(\sigma')$ is representable in spatial diamonds and that f has bounded topological transcendence degree. By ([51] 13.4) the first property can be checked pro-étale locally. Given a map from a spatial diamond $\mathcal{F} \to D(\sigma')$ we let $X = \mathcal{F}\times_{D(\sigma')} D(\sigma)$. Applying lemma 2.2.24 repeatedly to the quotients $L^+\mathcal{H}_{r_k}/L^+\mathcal{H}_{q_k}$ we get that pro-étale locally on $\mathcal{F}$, X is of the form $\mathcal{F}\times_{O_C^\diamondsuit} Fl^\diamondsuit_{r_n,q_n}$ which is a spatial diamond. Moreover, if $\mathcal{F} = \mathrm{Spa}(C', C'^+)$ with C' algebraically closed non-Archimedean field and C'^+ an open and bounded valuation subring then $X = Fl^\diamondsuit_{r_n,q_n,C'}$ and $dim.trg.(f) = dim(Fl_{r_n,q_n}) < \infty$ (See [51] 21.7). $\square$

Proposition 2.2.27. *Fix σ as above. The projection map $\pi : W^+\mathcal{H}_{r_1}\times_{O_C^\diamondsuit}\cdots\times_{O_C^\diamondsuit} W^+\mathcal{H}_{r_n} \to D(\sigma)$ induced from the family of injections $W^+\mathcal{H}_{r_i} \subseteq L^+\mathcal{H}_{r_i}$ is a surjective map of v-sheaves. It induces an identification:*

$$\iota : D(\sigma) \cong W^+\mathcal{H}_{r_1} \overset{W^+\mathcal{H}_{q_1}}{\times_{O_C^\diamondsuit}} \dots \overset{W^+\mathcal{H}_{q_{n-1}}}{\times_{O_C^\diamondsuit}} W^+\mathcal{H}_{r_n}/W^+\mathcal{H}_{q_n}$$

Consequently, $D(\sigma)$ is v-formalizing.

Proof. Consider the following basechange diagram:

$$\begin{tikzcd} W^+\mathcal{H}_{r_1}\times_{O_C^\diamond}\cdots\times_{O_C^\diamond}(L^+\mathcal{H}_{r_n}/L^+\mathcal{H}_{q_n}) \arrow[r] \arrow[d] & L^+\mathcal{H}_{r_1}\times_{O_C^\diamond}\cdots\times_{O_C^\diamond}(L^+\mathcal{H}_{r_n}/L^+\mathcal{H}_{q_n}) \arrow[r] \arrow[d] & D(\sigma) \arrow[d] \\ W^+\mathcal{H}_{r_1}\times_{O_C^\diamond}\cdots\times_{O_C^\diamond}W^+\mathcal{H}_{r_{n-1}} \arrow[r] & L^+\mathcal{H}_{r_1}\times_{O_C^\diamond}\cdots\times_{O_C^\diamond}L^+\mathcal{H}_{r_{n-1}} \arrow[r] & D(\sigma') \end{tikzcd} \tag{2.8}$$

Proposition 2.2.23 gives the equality $W^+\mathcal{H}_{r_n}/W^+\mathcal{H}_{q_n} = L^+\mathcal{H}_{r_n}/L^+\mathcal{H}_{q_n}$ and allow us to conclude surjectivity by induction. Assume that we have an identification:

$$\iota' : D(\sigma') \cong W^+\mathcal{H}_{r_1} \overset{W^+\mathcal{H}_{q_1}}{\times_{O_C^\diamond}} \dots \overset{W^+\mathcal{H}_{q_{n-2}}}{\times_{O_C^\diamond}} W^+\mathcal{H}_{r_{n-1}}/W^+\mathcal{H}_{q_{n-1}}$$

Since $W^+\mathcal{H}_{q_k} \subseteq L^+\mathcal{H}_{q_k}$ the map ι is defined and surjective, we need to prove that ι is also injective. Let $[g_1]$ and $[g_2]$ be two maps

$$[g_1],[g_2] : \mathrm{Spa}(R,R^+) \to W^+\mathcal{H}_{r_1} \overset{W^+\mathcal{H}_{q_1}}{\times_{O_C^\diamond}} \dots \overset{W^+\mathcal{H}_{q_{n-1}}}{\times_{O_C^\diamond}} W^+\mathcal{H}_{r_n}/W^+\mathcal{H}_{q_n}$$

and suppose that they get identified after mapping to $D(\sigma)(R,R^+)$. By our inductive hypothesis on $D(\sigma')$ we may locally for the v-topology find representatives g_1 and g_2 of $[g_1]$ and $[g_2]$ whose projection to the first $n-1$ coordinates is the same. That is g_i are of the form $(g_i^1,\dots,g_i^n)$ in $W^+\mathcal{H}_{r_1}\times_{O_C^\diamond}\cdots\times_{O_C^\diamond}W^+\mathcal{H}_{r_n}$ with $g_1^j = g_2^j$ for $j\in\{1,\dots,n-1\}$. Since $[g_1]$ and $[g_2]$ get identified in $D(\sigma)$ we must have that (v-locally) g_1 and g_2 are on the same $L^+\mathcal{H}_{q_1}\times_{O_C^\diamond}\cdots\times_{O_C^\diamond}L^+\mathcal{H}_{q_n}$-orbit. Since g_1 and g_2 share all of their entries except possibly the last we have that g_1^n and g_2^n are in the same $L^+\mathcal{H}_{q_n}$-orbit. Since $g_1^n, g_2^n\in W^+\mathcal{H}_{r_n}$ and $W^+\mathcal{H}_{q_n} = W^+\mathcal{H}_{r_n}\cap L^+\mathcal{H}_{q_n}$ they are in the same $W^+\mathcal{H}_{q_n}$-orbit which proves $[g_1]=[g_2]$.

Finally, by proposition 2.2.19 each $W^+\mathcal{H}_{r_i}$ is formalizing, proposition 1.4.11 implies the same for the product, and since $D(\sigma)$ is the quotient of a v-formalizing sheaf it is also v-formalizing. □

Lemma 2.2.28. *Given two points $r_1, r_2\in\mathcal{A}$ with $F_{r_1}\subseteq F_{r_2}$ the projection map of perfect schemes $\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{r_1}\to\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{r_1}/\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{r_2} = Fl^{perf}_{r_1,r_2}$ admits Zariski locally a section.*

Proof. Let $\mathfrak{obs}\in H^1_{v-Sch}(Fl^{perf}_{r_1,r_2},\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{r_2})$ be the obstruction to finding a section. Consider the reduction morphism exact sequence:

$$e\to\mathcal{WH}^u_{r_1}\to\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{r_2}\to P^{perf}_{\Phi_{r_1,r_2}}\to e$$

We also have $Fl^{perf}_{r_1,r_2} = M^{perf}_{r_1}/P^{perf}_{\Phi_{r_1,r_2}}$ and the cohomology class in $H^1_{v-Sch}(Fl^{perf}_{r_1,r_2},P^{perf}_{\Phi_{r_1,r_2}})$ associated to the $P^{perf}_{\Phi_{r_1,r_2}}$-torsor $M^{perf}_{r_1}\to Fl^{perf}_{r_1,r_2}$ is the image of $\mathfrak{obs}$. Zariski locally on $Fl^{perf}_{r_1,r_2}$ the $P^{perf}_{\Phi_{r_1,r_2}}$-torsor is trivial. This is known for the classical flag variety Fl_{r_1,r_2} over $\mathrm{Spec}(k_C)$ and the result will follow from taking perfection. Indeed, consider a commutative diagram trivializing the $P_{\Phi_{r_1,r_2}}$-torsor:

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\iota"] \arrow[dr, "s"'] & Fl_{p_1,p_2} \\ & M_{p_1} \arrow[u] \end{tikzcd}$$

Any such diagram having ι as Zariski open cover will produce a similar diagram after we take perfections and will trivialize the $P^{perf}_{\Phi_{r_1,r_2}}$-torsor.

Fix an affine cover $\mathrm{Spec}(R) \to Fl^{perf}_{r_1,r_2}$ trivializing $red(\mathfrak{obs})$. We claim that $\mathrm{Spec}(R)$ has a section to $\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{r_1} \times_{Fl^{perf}_{r_1,r_2}} \mathrm{Spec}(R)$. By construction, we know that

$$\mathfrak{obs} \in H^1_{v-Sch}(\mathrm{Spec}(R), \mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{r_2})$$

is in the image of $H^1_{v-Sch}(\mathrm{Spec}(R), \mathcal{W}^u\mathcal{H}_{r_1})$, but this pointed set is trivial. Indeed, we have that $H^1_{v-Sch}(\mathrm{Spec}(R), \mathbb{G}_a) = \{e\}$ which is a particular case of theorem 4.1 in [5]. One can finish the proof by using the argument given in lemma 2.2.24. □

Proposition 2.2.29. *The map $D(\sigma) \to O_C^\lozenge$ is formally adic. Moreover, $D(\sigma)^{\mathrm{red}}$ is represented by a qcqs scheme that is perfectly finitely presented and proper over* $\mathrm{Spec}(k_C)$ *(See [5] 3.11 and 3.14 for definitions).*

Proof. Let $\sigma = (\{r_i\}_{1\leq i\leq n}, \{q_i\}_{1\leq i\leq n})$ and let $\sigma' = (\{r_i\}_{1\leq i\leq n-1}, \{q_i\}_{1\leq i\leq n-1})$. In any topos pullback commutes with finite limits and colimits, so by proposition 2.2.19 we have:

$$D(\sigma) \times_{O_C^\lozenge} \mathrm{Spec}(k_C)^\lozenge = (\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{r_1})^\lozenge \overset{(\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{q_1})^\lozenge}{\times_{k_C^\lozenge}} \cdots \overset{(\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{q_{n-1}})^\lozenge}{\times_{k_C^\lozenge}} (\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{r_n})^\lozenge / (\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{q_n})^\lozenge$$

Since the functor $(\cdot)^\lozenge$ is a left adjoint it commutes with colimits, so we get:

$$D(\sigma) \times_{O_C^\lozenge} \mathrm{Spec}(k_C)^\lozenge = \left(\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{p_1} \overset{\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{q_1}}{\times_{k_C}} \cdots \overset{\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{q_{n-1}}}{\times_{k_C}} \mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{p_n} / \mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{q_n} \right)^\lozenge$$

Lemma 1.3.35 proves that $D(\sigma) \to O_C^\lozenge$ is formally adic and that $D(\sigma)^{\mathrm{red}} = D(\sigma) \times_{O_C^\lozenge} \mathrm{Spec}(k_C)^\lozenge$. In particular $\mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{r_k} / \mathcal{W}^+_{\mathrm{red}}\mathcal{H}_{q_k} = Fl_{r_k,q_k}$.

We prove inductively that $D(\sigma)^{\mathrm{red}}$ is represented by a qcqs scheme perfectly finitely presented and proper over $\mathrm{Spec}(k_C)$. Iterating lemma 2.2.28 we see that the map $D(\sigma)^{\mathrm{red}} \to D(\sigma')^{\mathrm{red}}$ is Zariski locally a trivial $Fl^{perf}_{r_k,q_k}$-bundle. Now, quasi-compactness, separatedness, quasi-separatedness, representability in schemes, being perfectly of finite presentation and being proper can all be checked Zariski locally on the target and are stable under basechange and composition (See [56] Tag 02YJ). By induction, $D(\sigma')$ enjoys all of these properties over $\mathrm{Spec}(k)$ and $Fl_{r_n,q_n} \times_{k_C} \mathrm{Spec}(A)$ enjoys them over $\mathrm{Spec}(A)$ for any affine open $\mathrm{Spec}(A) \subseteq D(\sigma')^{\mathrm{red}}$. This proves that $D(\sigma)$ also enjoys them over $\mathrm{Spec}(k_C)$, which finishes the proof. □

Proposition 2.2.30. *For any σ and any geometric point* $\mathrm{Spa}(C', O_{C'}) \to O_C^\lozenge$ *the base change* $D(\sigma) \times_{O_C^\lozenge} \mathrm{Spa}(C', O_{C'})$ *is a cJ-diamond.*

Proof. We prove by induction that $D(\sigma)$ has enough facets, let $\sigma = (\{r_i\}_{1\leq i\leq n}, \{q_i\}_{1\leq i\leq n})$ and let $\sigma' = (\{r_i\}_{1\leq i\leq n-1}, \{q_i\}_{1\leq i\leq n-1})$. Suppose that $D(\sigma')_{C'}$ has enough facets, let $S := \coprod_{i\in I} \mathrm{Spd}(B_i, B_i^\circ)$ with each B_i a topologically of finite type C'-algebra and let $f : S \to D(\sigma')_{C'}$ be a surjective map. Let $\mathcal{F} = D(\sigma)_{C'} \times_{D(\sigma')_{C'}} S$, we prove that $\mathcal{F}$ has a enough facets. Analytically locally on S the projection map $\mathcal{F} \to S$ is a trivial $(Fl_{r_n,q_n,C'^\sharp})^\lozenge$-fibration. The proof of this claim is an iteration of the argument given on lemma 2.2.24 together with the observation that S is already a disjoint union of affinoid perfectoid spaces. We may replace S by an analytic cover S' so that we get the expression:

$$\mathcal{F}' = D(\sigma)_{C'} \times_{D(\sigma')_{C'}} S' = \coprod \mathrm{Spd}(B_i', (B_i')^\circ) \times_{Spa(C',O_{C'})} (Fl_{r_n,q_n,C'^\sharp})^\lozenge$$

By proposition 1.4.39 having enough facets is stable under products so $\mathcal{F}'$ has enough facets and consequently $D(\sigma)$ as well. $\square$

We can summarize this subsection with the following theorem:

Theorem 2.2.31. *For any σ as in definition 2.2.25 the Demazure kimberlite $D(\sigma)$ is a rich p-adic kimberlite. The p-adic tubular neighborhoods are non-empty and connected, and the structure morphism $D(\sigma) \to O_C^\lozenge$ is proper and ℓ-cohomologically smooth.*

Proof. Separatedness and the properties of the structure morphism were proven on proposition 2.2.26. That it is v-formalizing is proven in proposition 2.2.27. We have

$$(D(\sigma)^{\mathrm{red}})^\lozenge = D(\sigma) \times_{O_C^\lozenge} \mathrm{Spec}(k_C)^\lozenge$$

this implies that the adjunction morphism $(D(\sigma)^{\mathrm{red}})^\lozenge \to D(\sigma)$ is a closed embedding. By proposition 1.3.31 $D(\sigma)$ is formally separated and specializing. By proposition 2.2.29 $D(\sigma)^{\mathrm{red}}$ is represented by a scheme. At this point we have proved that $D(\sigma)$ is a p-adic pre-kimberlite.

Since $D(\sigma)^{\mathrm{red}}$ is represented by a proper perfectly finitely presented scheme over k_C then $|D(\sigma)^{\mathrm{red}}|$ is a Noetherian and spectral topological space. The analytic locus $D(\sigma)^{an}$ coincides with the generic fiber $D(\sigma) \times_{O_C^\lozenge} \mathrm{Spa}(C, O_C)$ and by proposition 2.2.30 this is a cJ-diamond. One can easily prove inductively over the map $D(\sigma) \to D(\sigma')$ that the specialization map is surjective on closed points. By lemmas 1.4.43 and 1.4.44 it is a quotient map. This finishes the proof that $D(\sigma)$ is a rich kimberlite.

The connectedness of tubular neighborhoods will follow from lemma 2.2.32 below. Indeed, we have already verified that all but conditions 4 and 5 of this lemma hold. Condition 5 holds by induction over the maps $D(\sigma) \to D(\sigma')$ and condition 4 follows from the diagram 2.8 since each of the $W^+\mathcal{H}_{r_i}$ is formalizing and basechanges along maps that factor through $W^+\mathcal{H}_{r_1} \times_{O_C^\lozenge} \cdots \times_{O_C^\lozenge} W^+\mathcal{H}_{r_{n-1}}$ will give a trivial bundle. $\square$

Lemma 2.2.32. *Let $f : \mathcal{F} \to \mathcal{G}$ be a map of kimberlites over $O_C^\lozenge$, let $X \to \mathrm{Spec}(O_{C^\sharp})$ be a smooth projective scheme. Suppose the following properties hold:*

1. *f is ℓ-cohomologically smooth for some $\ell \neq p$.*

2. *f is proper.*

3. *$f : \mathcal{F} \to \mathcal{G}$ is a $X(\mathcal{O}^{\sharp,+})$-bundle locally trivial for the v-topology.*

4. *For any non-Archimedean field C' in characteristic 0 and any map $t : Spa(C', O_{C'}) \to \mathcal{G}$ there is a v-cover $r : Spa(C'', O_{C''}) \to Spa(C', O_{C'})$ such that $\mathcal{G}$ formalizes $t \circ r$ and the base change $\mathcal{F} \times_{\mathcal{G}} O^{\Diamond}_{C''}$ is isomorphic to $X(\mathcal{O}^{\sharp,+}) \times_{O^{\Diamond}_{C}} O^{\Diamond}_{C''}$.*

5. *For any closed point $x \in |\mathcal{G}^{\mathrm{red}}|$ the p-adic tubular neighborhood $(\widehat{\mathcal{G}}_{/x})_\eta$ is connected.*

6. *$\mathcal{G}^{\mathrm{red}}$ and $\mathcal{F}^{\mathrm{red}}$ are perfectly finitely presented (See [5] 3.10) over $\mathrm{Spec}(k_C)$.*

Then, for any closed point $y \in |\mathcal{F}^{\mathrm{red}}|$ the p-adic tubular neighborhood $(\widehat{\mathcal{F}}_{/y})_\eta$ is also connected.

Proof. We observe that f is open and closed since it is ℓ-cohomologically smooth and proper (See [51] 23.11). Take a closed point $y \in |\mathcal{F}^{\mathrm{red}}|$ with $x = f(y)$ and consider the map $f : (\widehat{\mathcal{F}}_{/y})_\eta \to (\widehat{\mathcal{G}}_{/x})_\eta$. Assume for the moment that given C' an arbitrary algebraically closed non-Archimedean field and a map of the form $\mathrm{Spa}(C', O_{C'}) \to (\widehat{\mathcal{G}}_{/x})_\eta$ the base change $(\widehat{\mathcal{F}}_{/y})_\eta \times_{(\widehat{\mathcal{G}}_{/x})_\eta} \mathrm{Spa}(C', O_{C'})$ is always non-empty and connected, we finish the proof under this assumption. Observe that the map of topological spaces $|(\widehat{\mathcal{F}}_{/y})_\eta| \to |(\widehat{\mathcal{G}}_{/x})_\eta|$ is specializing, and by assumption surjective on rank 1 points. Take two non-empty open and closed subsets U and V with $U \cup V = |(\widehat{\mathcal{F}}_{/y})_\eta|$. Then $f(U) \cup f(V) = |(\widehat{\mathcal{G}}_{/x})_\eta|$ and consequently $f(U) \cap f(V) \neq \emptyset$. Since f is an open map $f(U)$ and $f(V)$ must meet in a rank 1 point, this implies that U and V also meet which finishes the proof under our assumption.

Let us prove our assumption holds. Take a map $t : \mathrm{Spa}(C', O_{C'}) \to (\widehat{\mathcal{G}}_{/x})_\eta$ and after replacing $\mathrm{Spa}(C', O_{C'})$ by a v-cover we can assume $\mathcal{G}$ formalizes the composition $\mathrm{Spa}(C', O_{C'}) \to \mathcal{G}$ and has the base change property of condition 4 with respect to the unique formalization $O^{\Diamond}_{C'} \to \mathcal{G}$. We get a Cartesian diagram:

$$\begin{tikzcd}
\widehat{\mathcal{F}}_{/y} \times_{\mathcal{G}} O^{\Diamond}_{C'} \arrow[r] \arrow[d] & X(\mathcal{O}^{\sharp,+}) \times_{O^{\Diamond}_{C}} O^{\Diamond}_{C'} \arrow[r] \arrow[d] & O^{\Diamond}_{C'} \arrow[d] \\
\widehat{\mathcal{F}}_{/y} \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{G}
\end{tikzcd}$$

After taking reduction functor of this diagram we get the following Cartesian diagram:

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & X \times \mathrm{Spec}(k') \arrow[r] \arrow[d] & \mathrm{Spec}(k') \arrow[d] \\
\mathrm{Spec}(k(y)) \arrow[r, "y"] & \mathcal{F}^{\mathrm{red}} \arrow[r] & \mathcal{G}^{\mathrm{red}}
\end{tikzcd}$$

Since $\mathcal{F}^{\mathrm{red}} \to \mathcal{G}^{\mathrm{red}}$ is perfectly finitely presented and k is algebraically closed we have that $k = k(y) = k(x)$ and the composition $y : \mathrm{Spec}(k) \to \mathcal{G}^{\mathrm{red}}$ is the closed immersion corresponding to the point x. Consequently $Z \to \mathrm{Spec}(k')$ is a closed immersion, therefore an isomorphism. We have that

$$\widehat{\mathcal{F}}_{/y} \times_{\mathcal{G}} O_{C'}^{\Diamond} = X(\mathcal{O}^{\sharp,+})_{O_{C'}} \times_{\mathcal{F}} \widehat{\mathcal{F}}_{/y} = (X(\widehat{\mathcal{O}^{\sharp,+})}_{O_{C'}})_{/Z}$$

by proposition 1.4.26. But $Z \to X \times \mathrm{Spec}(k')$ is a closed point, so $(X(\widehat{\mathcal{O}^{\sharp,+})}_{O_{C'}/Z})_{\eta}$ is isomorphic to an open unit ball $\mathbb{B}_n^{<1}$ over $C'^{\sharp}$, where $n = dim(X)$ and $C'^{\sharp}$ is the untilt determined by $O_{C^{\sharp}}$. We have proved that the fibers are non-empty and connected. □

2.2.4 Resolution of p-adic Beilinson-Drinfeld Grassmanians

In this subsection we discuss an analogue of the Demazure resolution for split reductive groups in the context of v-sheaves (also known as Bott-Samelson resolution). We keep the notation from the beginning of the previous subsection and we restrict our attention to parahoric loop groups associated to points contained in our chosen alcove $\mathcal{C}$. Given $s_j \in \mathbb{S}$ we denote by $L^+\mathcal{H}_{s_j}$ the parahoric loop group associated to the wall F_{s_j} in $\mathcal{C}$ corresponding to the reflection s_j. For a point $r \in \mathcal{C}$ we let $J_r \subseteq \mathbb{S}$ denote the set $\{s_j \mid r \in F_{s_j}\}$. We will denote by L^+B the parahoric loop group associated $\mathcal{C}$.

By functoriality of $L(-)$ we can define a loop group version of the Weyl group by the formula $LW := LN/LT$. We can also define the Iwahori-Weyl group as $L\tilde{W} := LN/L^+\mathcal{T}$. There is an exact sequence of v-sheaves in groups:

$$e \to LT/L^+\mathcal{T} \to L\tilde{W} \to LW \to e$$

One can prove by a direct computation that $LW = L(N/T) = \underline{W} \times O_C^{\Diamond}$ and that $LT/L^+\mathcal{T} = \underline{X_*(T)} \times O_C^{\Diamond}$ by using the Cartan decomposition. These two imply that $L\tilde{W} = \underline{\tilde{W}} \times O_C^{\Diamond}$.

Since H is a split reductive group over $W(k)[t,t^{-1}]$, for any element $w \in W$ we can find a section $n_w : \mathrm{Spec}(W(k)[t,t^{-1}]) \to N$ whose projection to W is w (See [10] 5.1.11). This allow us to define a similar section $n_w : O_C^{\Diamond} \to LN \subseteq LH$. Also for any $\mu \in X_*(T)$ and any $\mathrm{Spa}(R,R^+) \to O_C^{\Diamond}$ we can consider the element $\xi^{\mu} \in T(B_{dR}(R^{\sharp}))$. This is functorial and defines a section $O_C^{\Diamond} \to LT$ mapping to $\mu \in \underline{X_*(T)} \times O_C^{\Diamond}$. In particular, for any element $\tilde{w} \in \tilde{W}$ there is a section $n_{\tilde{w}} : O_C^{\Diamond} \to LN$ projecting to $\tilde{w}$ in $L\tilde{W}$. We can use $n_{\tilde{w}}$ to construct an automorphism $n_{\tilde{w}} : \mathrm{Gr}^H_{O_C^{\Diamond}} \to \mathrm{Gr}^H_{O_C^{\Diamond}}$ with

$$n_{\tilde{w}}(x \cdot L^+H) := n_{\tilde{w}} \cdot x \cdot L^+H.$$

We will use this discussion in the proof of theorem 2.2.34.

Proposition 2.2.33. *Let $\sigma = (\sigma_r, \sigma_q)$ with σ as in the previous subsection except that we require $\sigma_r, \sigma_q \subseteq \mathcal{C}$. Suppose that $L^+\mathcal{H}_{q_n} = L^+\mathcal{H}_{r_n} = L^+H$ then the multiplication map $\mu : D(\sigma) \to \mathrm{Gr}^H_{O_C} = LH/L^+H$ has geometrically connected fibers.*

Proof. This proof follows the classical one. The key inputs are as follows, the basechange of $D(\sigma) \to O_C^{\Diamond}$ by geometric points are proper spatial diamonds, rank 1 points are dense for any spatial diamond and the group of rank 1 geometric points of a parahoric loop group coincide with the "parabolic subgroups" of a Tits-systems (or BN-pair). These two observations together with ([51] 12.11) reduces the proof to the classical combinatorial case. Indeed, properness (which includes quasi-compactness) will allow us to prove all surjectivity claims at the level of rank 1 geometric points. We provide further details below for the convenience of the reader.

Fix a geometric point $\mathrm{Spa}(C', O_{C'}) \to O_C^\lozenge$, all of the objects considered in our argument below are considered over $\mathrm{Spa}(C', O_{C'})$ but we omit the basechange from the notation. Let us start by making some reductions, observe that since we are assuming that $\sigma_q \subseteq \mathcal{C}$ we have $L^+B \subseteq L^+\mathcal{H}_{q_i}$ so we get a surjective map:

$$D(\tau) := L^+\mathcal{H}_{r_1} \overset{L^+B}{\times} \dots \overset{L^+B}{\times} L^+\mathcal{H}_{r_n}/L^+B \to D(\sigma)$$

Surjectivity allows us to replace $D(\sigma)$ for $D(\tau)$ so we may assume $L^+\mathcal{H}_{q_i} = L^+B$ for all $i \leq n$. Now the flag varieties $L^+\mathcal{H}_{r_i,}/L^+B$ admit a surjective map from a finite contracted product of the form:

$$L^+\mathcal{H}_{s_{j_1},} \overset{L^+B}{\times} \dots \overset{L^+B}{\times} L^+\mathcal{H}_{s_{j_m},}/L^+B \to L^+\mathcal{H}_{r_1,}/L^+B$$

Where $s_{j_k} \in J_{r_i}$ and the product $s_{j_1} \cdot \dots \cdot s_{j_m}$ is a reduced expression for the longest word in the finite Coxeter group generated by J_{r_i}. This lets us reduce to the case in which for all $i \leq n$, $L^+\mathcal{H}_{r_i} = L^+\mathcal{H}_{s_j}$ for some j. Moreover, in this case the map $D(\tau) \to \mathrm{Gr}^H_{C'}$ factors through $LH/L^+B \to \mathrm{Gr}^H_{C'}$ which is a L^+H/L^+B-bundle.

We prove inductively that $D(\tau) \to LH/L^+B$ has connected geometric fibers. Write $S(\tau)$ for the image of $D(\tau)$ in LH/L^+B. The multiplication map factors as:

$$D(\tau) = L^+\mathcal{H}_{s_{j_1},} \overset{L^+B}{\times} D(\tau') \to L^+\mathcal{H}_{s_{j_1},} \overset{L^+B}{\times} S(\tau') \to S(\tau)$$

If we assume inductively that the map $D(\tau') \to S(\tau')$ has connected geometric fibers, then it suffices to prove that $L^+\mathcal{H}_{s_{j_1},} \overset{L^+B}{\times} S(\tau') \to S(\tau)$ also has connected geometric fibers.

Notice that by construction $S(\tau') \subseteq LH/L^+B$ is a closed subsheaf that is stable under the action of L^+B. As in the classical case the L^+B-orbits of geometric points in LH/L^+B are indexed $\tilde{W}$. Given an element $w \in \tilde{W}$ we can consider $C(w)$ the locally-closed subsheaf of LH/L^+B associated to this L^+B-orbit and we can let $S(w) = \bigcup_{w' \leq w} C(w')$ where '$\leq$' denotes the Bruhat order of the quasi-Coxeter system $\mathbb{S} \subseteq W^{aff} \subseteq \tilde{W}$. We also assume in our inductive hypothesis that $S(\tau') = S(w)$ for $w \in W^{aff}$ of the form $w = s_{j_{k_1}} \dots s_{j_{k_l}}$ where s_{j_k} is a subsequence of elements in $\mathbb{S}$ of the sequence appearing in the definition of $D(\tau')$.

For the induction step we have two cases, either $s_{j_1} \cdot w < w$ or $s_{j_1} \cdot w > w$. In the first case we will have that the action of $L^+\mathcal{H}_{s_{j_1}}$ on LH/L^+B stabilizes $S(w)$ so that the multiplication map

$$L^+\mathcal{H}_{s_{j_1}} \overset{L^+B}{\times} S(w) \to S(w)$$

decomposes as the composition of an isomorphism $L^+\mathcal{H}_{s_{j_1}} \overset{L^+B}{\times} S(w) \to L^+\mathcal{H}_{s_{j_1}}/L^+B \times S(w)$ followed by the second projection. In this case geometric fibers are isomorphic to $L^+\mathcal{H}_{s_{j_1},C''}/L^+B_{C''} \cong (\mathbb{P}^1_{C''^\sharp})^\lozenge$, and we have that $S(\tau) = S(\tau') = S(w)$ which is of the form assumed in our inductive hypothesis.

On the other hand, if $s_{j_1} \cdot w > w$ we can consider the collection T of $w' < w$ for which $s \cdot w' < w'$ we have that the multiplication map

$$L^+\mathcal{H}_{s_{j_1}} \overset{L^+B}{\times} S(w) \setminus \bigcup_{w' \in T} S(w') \to S(s_{j_1} \cdot w) \setminus \bigcup_{w' \in T} S(w')$$

is an isomorphism while the map

$$L^+\mathcal{H}_{s_{j_1}} \overset{L^+B}{\times} \bigcup_{w' \in T} S(w') \to \bigcup_{w' \in T} S(w')$$

has geometric fibers as in the previous case since this set is also $L^+\mathcal{H}_{s_{j_1}}$-stable. Moreover, we have $S(\tau) = S(s_{j_1} \cdot w)$ which is again of the form assumed in our induction hypothesis, this finishes inductive step and the proof. □

Theorem 2.2.34. *Let $\mathscr{G}$ be a quasi-split reductive group over $W(k)$, $\mathfrak{T} \subseteq \mathfrak{B} \subseteq \mathscr{G}$ a Borel and a maximal torus in $\mathscr{G}$ defined over $W(k)$ and take a cocharacter $\mu \in X_*(\mathfrak{T})$ defined over an algebraic closure of $W(k)[\frac{1}{p}]$. Let F be a non-Archimedean field extension of $W(k)[\frac{1}{p}]$ containing $E(\mu)$ the reflex field of μ. We let O_F the ring of integers of F and the residue field k_F, assume that F is complete for the p-adic topology and that k_F is perfect. Then $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_F^\lozenge}$ is an rich p-adic kimberlite over $O_F^\lozenge$. Moreover, the p-adic tubular neighborhoods of $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_F^\lozenge}$ at closed points are non-empty and connected.*

Proof. In corollary 2.2.7 we show that $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_F^\lozenge}$ is a p-adic kimberlite so the only thing left to prove are the statements related to the structure of p-adic tubular neighborhoods. We first prove the case in which F is a complete algebraically closed extension of $W(k)[\frac{1}{p}]$ which we will denote instead by C. In this case $\mathscr{G} \times_{W(k)} W(k_C)$ is isomorphic to a split reductive group, and since the functor $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_C^\lozenge}$ only depends on the isomorphism class of $\mathscr{G}_{W(k_C)}$, we may assume $\mathscr{G} = H$ with H split reductive. Furthermore, we discuss first the case in which H is semisimple and simply connected, in this case $\tilde{W} = W^{aff}$.

Recall that we have inclusions $X_*^+(\mathfrak{T}) \subseteq X_*(\mathfrak{T}) \subseteq \tilde{W}$ so we may think of μ as an element of the Iwahori-Weyl group. By definition, $\mathrm{Gr}^{H,\leq\mu}_{O_C^\lozenge}(R, R^+)$ consists of those elements in $\mathrm{Gr}^H_{O_C^\lozenge}(R, R^+)$ satisfying that for any geometric point $q : \mathrm{Spa}(C', C'^+) \to \mathrm{Spa}(R, R^+)$ the type of q, μ_q, is in the double coset

$$H(B^+_{dR}(C'^\sharp))\backslash H(B_{dR}(C'^\sharp))/H(B^+_{dR}(C'^\sharp)) = X^+_*(T) = W_o\backslash W^{aff}/W_o$$

satisfies that $\mu_q \leq \mu$ in the Bruhat order. Now given any element $w \in \tilde{W}$ we may consider the subsheaf $\mathrm{Gr}^{G,\leq w}_{O^\diamondsuit_C} \subseteq \mathrm{Gr}^{G}_{O^\diamondsuit_C}$ given instead by the property that on a geometric point $q : \mathrm{Spa}(C', C'^{+}) \to \mathrm{Spa}(R, R^+)$ the type of q, $[w_q]$, in the double coset

$$\mathfrak{B}(B^+_{dR}(C'^\sharp))\backslash H(B_{dR}(C'^\sharp))/H(B^+_{dR}(C'^\sharp)) = W^{aff}/W_o$$

satisfies that $[w_q] \leq [w]$ in the Bruhat order. The projection map $\pi : W^{aff}/W_o \to W_o\backslash W^{aff}/W_o$ is order preserving and $\pi^{-1}(\mu)$ has a unique element $[w_\mu]$ of largest length, it has the property that $v \leq w_\mu$ if and only if $\pi(v) \leq \mu$. In particular, we have equalities of sheaves $\mathrm{Gr}^{H,\leq w_\mu}_{O^\diamondsuit_C} = \mathrm{Gr}^{H,\leq \mu}_{O^\diamondsuit_C}$. We prove that for any word $w \in W^{aff}$ the v-sheaf $\mathrm{Gr}^{H,\leq w}_{O^\diamondsuit_C}$ satisfies the conclusions of the theorem. If we find a reduced expression for $w = s_{j_1} \dots s_{j_n}$ we can use the theory of BN-pairs to construct a Demazure kimberlite

$$D(w) := L^+H_{s_{j_1}} \overset{L^+\mathfrak{B}}{\times_{O^\diamondsuit_C}} \dots L^+H_{s_{j_n}}/L^+H$$

for which the multiplication map $m : D(w) \to \mathrm{Gr}^{H}_{O^\diamondsuit_C}$ factors through $\mathrm{Gr}^{H,\leq w}_{O^\diamondsuit_C}$ and surjects onto it at the level of rank 1 geometric points. But m is a proper map so that by ([51] 12.11) it is actually a surjection of v-sheaves. Moreover, this also proves that $\mathrm{Gr}^{H,\leq w}_{O^\diamondsuit_C}$ is a closed subsheaf of $\mathrm{Gr}^{H}_{O^\diamondsuit_C}$. Theorem 2.2.31 and proposition 2.2.33 combined with lemma 1.4.45 allow us to conclude in this case.

Suppose now that H is an arbitrary split reductive group. In this case, $\mu \in \tilde{W}$ can be expressed as $\mu = (w, \omega)$ with $w \in W^{aff}$ and $\omega \in \Omega_H$ for the decomposition $\tilde{W} = W^{aff} \rtimes \Omega_H$. We may find a section $n_\omega : O^\diamondsuit_C \to LN$ projecting to (e, ω) in $L\tilde{W}$, this section induces an isomorphism between $\mathrm{Gr}^{H,\leq \mu}_{O^\diamondsuit_C}$ and $\mathrm{Gr}^{H,\leq (w,e)}_{O^\diamondsuit_C}$. But for any $w \in W^{aff}$ the v-sheaf $\mathrm{Gr}^{H,\leq (w,e)}_{O^\diamondsuit_C}$ admits a surjective map by a Demazure kimberlite as in the previous case.

Finally, let us deal with the general case in which F is not assumed to be algebraically closed. Let C be the completion of an algebraic closure of F and F' the completion of the maximal unramified subextension of F inside C. We have surjective maps of v-sheaves:

$$\mathrm{Gr}^{\mathscr{G},\leq \mu}_{O^\diamondsuit_C} \to \mathrm{Gr}^{\mathscr{G},\leq \mu}_{O^\diamondsuit_{F'}} \to \mathrm{Gr}^{\mathscr{G},\leq \mu}_{O^\diamondsuit_F}$$

Lemma 1.4.45 implies that $\mathrm{Gr}^{\mathscr{G},\leq \mu}_{O^\diamondsuit_F}$ and $\mathrm{Gr}^{\mathscr{G},\leq \mu}_{O^\diamondsuit_{F'}}$ are rich kimberlites. Moreover, we can infer by proposition 1.4.26 that $\mathrm{Gr}^{\mathscr{G},\leq \mu}_{O^\diamondsuit_{F'}}$ has connected p-adic tubular neighborhoods since we have an identification $(\mathrm{Gr}^{\mathscr{G},\leq \mu}_{O^\diamondsuit_C})^{\mathrm{red}} = (\mathrm{Gr}^{\mathscr{G},\leq \mu}_{O^\diamondsuit_{F'}})^{\mathrm{red}}$. On the other hand, the map $\mathrm{Gr}^{G,\leq \mu}_{O^\diamondsuit_{F'}} \to \mathrm{Gr}^{\mathscr{G},\leq \mu}_{O^\diamondsuit_F}$ is a $\underline{\pi_1}^{f\acute{e}t}(\mathrm{Spec}(O_F))$-torsor and for any closed point $x \in |\mathrm{Gr}^{\mathscr{G},\leq \mu^{\mathrm{red}}}_{O^\diamondsuit_F}|$ the action of $\pi_1^{f\acute{e}t}(\mathrm{Spec}(O_F))$ will permute transitively the closed points $y \in |\mathrm{Gr}^{\mathscr{G},\leq \mu^{\mathrm{red}}}_{O^\diamondsuit_{F'}}|$ over x. In particular, the action permutes transitively the p-adic tubular neighborhoods associated to such y. This proves that the tubular neighborhood over x is also connected. $\square$

We finish this section with the proof of theorem 1 which is just a rephrasing of theorem

2.2.34 in less technical language. For the convenience of the reader we write the statement again.

Theorem 2.2.35. *With notation as in the introduction the following holds:*

a) The specialization map

$$\mathrm{sp}_{\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O^{\diamondsuit}_{F_1}}} : |\mathrm{Gr}^{G,\leq\mu}_{F_1{}^{\diamondsuit}}| \to |\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}|$$

is a closed and spectral map of spectral topological spaces.

b) Given a closed point $x \in |\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}|$ let $T_x := \mathrm{sp}_{\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O^{\diamondsuit}_{F_1}}}{}^{-1}(x)$, then the interior T_x° of T_x in $|\mathrm{Gr}^{G,\leq\mu}_{F_1{}^{\diamondsuit}}|$ is a dense subset of T_x.

c) T_x and T_x° are connected.

Proof of theorem 1. We may apply theorem 2.2.34 and proposition 2.2.5 to the case in which $k = \mathbb{F}_p$ to conclude that $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O^{\diamondsuit}_{F_1}}$ is a rich p-adic kimberlite with generic fiber $\mathrm{Gr}^{G,\leq\mu}_{F_1{}^{\diamondsuit}}$ and with reduction $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}$. Since $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O^{\diamondsuit}_{F_1}}$ is a kimberlite by proposition 1.4.20 the specialization map

$$\mathrm{sp}_{\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O^{\diamondsuit}_{F_1}}} : |\mathrm{Gr}^{G,\leq\mu}_{F_1{}^{\diamondsuit}}| \to |\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}|$$

is a spectral map of locally spectral spaces. Since $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O^{\diamondsuit}_{F_1}}$ is rich, the map is surjective and specializing, and since $\mathrm{Gr}^{G,\leq\mu}_{F_1{}^{\diamondsuit}}$ is quasi-compact proposition 1.1.22 gives that the specialization map is closed, this finishes the proof of the first claim. For the second claim let $x \in |\mathrm{Gr}^{\mathscr{G},\leq\mu}_{\mathcal{W},k_{F_1}}|$, we can use proposition 1.4.29 to identify T_x° with $|(\widehat{\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_{F_1}}}_{/x})_\eta|$. Since $\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_{F_1}}$ is rich we can apply proposition 1.4.33 to prove that T_x° is dense in T_x giving the second claim. By 2.2.34 T_x° is connected and since it is dense in T_x this later one is also connected. □

2.3 Specialization for moduli of mixed characteristic shtukas

For the rest of this section we will assume that $k = \mathbb{F}_p$ and that $\mathscr{G}$ is a reductive group over $\mathbb{Z}_p$. We fix a torus and a Borel $\mathfrak{T} \subseteq \mathfrak{B} \subseteq \mathscr{G}$. We fix $\mathfrak{f}$ an algebraically closed field extension of $\overline{\mathbb{F}}_p$ and we let $K_0 = W(\mathfrak{f})[\frac{1}{p}]$, we fix an element $b \in \mathscr{G}(K_0)$ and we let $\mathscr{G}_b : \mathrm{Rep}^{\mathscr{G}}_{\mathbb{Z}_p} \to \mathrm{IsoCrys}_{K_0}$ denote the $\otimes$-exact functor from the category of algebraic representations of $\mathscr{G}$ to the category of isocrystals over K_0 associated to b.

Definition 2.3.1. *We define the moduli space of mixed characteristic shtukas associated to $\mathscr{G}_b$, which we denote by $\mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$, as the functor $\mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})} : \mathrm{Perf} \to \mathrm{Sets}$:*

$$\mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}(R, R^+) = \{(R^\sharp, \iota, f), \mathscr{T}, \Phi, \lambda\}/\cong$$

Where $(R^\sharp, \iota, f)$ *denotes an untilt of* R *over* $\mathrm{Spa}(W(\mathfrak{f}), W(\mathfrak{f}))$*, the pair* $(\mathscr{T}, \Phi)$ *is a shtuka as in definition 2.1.22 and* $\lambda : \mathscr{T} \to \mathscr{G}_b|_{\mathcal{Y}^{R^+}_{[r,\infty)}}$ *is an equivalence class of isogenies as in definition 2.1.23. Here* $\mathscr{G}_b|_{\mathcal{Y}^{R^+}_{[r,\infty)}}$ *denotes the pullback along the natural map of locally ringed spaces* $\mathcal{Y}^{R^+}_{[r,\infty)} \to \mathrm{Spec}(K_0)$ *induced by* f*.*

As with p-adic Beilinson-Drinfeld Grassmanians, moduli spaces of shtukas admit bounded versions. Given a geometric point of our moduli $\mathrm{Spa}(C, C^+) \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ the torsor $\mathscr{T}$ can be glued with $\mathscr{G}_b$ along $\mathcal{Y}^{C^+}_{[0,\infty)}$ to extend it to $\mathcal{Y}_{C^+}$. This gives a $\mathscr{G}$-torsor over Y_{C^+} by theorem 2.1.12. One can basechange this torsor to $B^+_{dR}(C^\sharp)$ where we can choose a trivialization of $\tau : \mathscr{T} \to \mathscr{G}$. The morphism $\tau \circ \Phi : \phi^* \mathscr{T} \to \mathscr{G}$ defines an element of $\mathscr{G}(B_{dR}(C^\sharp))/\mathscr{G}(B^+_{dR}(C^\sharp))$ whose image, $\mu_{(\mathscr{T},\Phi)}$, in the double coset

$$\mathscr{G}(B^+_{dR}(C^\sharp)) \backslash \mathscr{G}(B_{dR}(C^\sharp))/\mathscr{G}(B^+_{dR}(C^\sharp)) = X^+_*(\mathfrak{T}_{\overline{\mathbb{Q}}_p})$$

does not depend on the choice of τ. We call $\mu_{(\mathscr{T},\Phi)}$ the relative position of the shtuka at that geometric point.

Definition 2.3.2. *Let* $\mu \in X^+_*(\mathfrak{T}_{\overline{\mathbb{Q}}_p})$*. We define the moduli space of mixed characteristic shtukas associated to* $\mathscr{G}_b$ *and bounded by* μ*, which we denote by* $\mathrm{Sht}^{\mathscr{G}_b, \leq \mu}_{W(\mathfrak{f})}$*, as the functor* $\mathrm{Sht}^{\mathscr{G}_b, \leq \mu}_{W(\mathfrak{f})} : \mathrm{Perf} \to \mathrm{Sets}$*:*

$$\mathrm{Sht}^{\mathscr{G}_b, \leq \mu}_{W(\mathfrak{f})}(R, R^+) = \{(R^\sharp, \iota, f), \mathscr{T}, \Phi, \lambda\}/\cong$$

Where $(R^\sharp, \iota, f)$ *denotes an untilt of* R *over* $\mathrm{Spa}(W(\mathfrak{f}), W(\mathfrak{f}))$*, the pair* $(\mathscr{T}, \Phi)$ *is a shtuka whose relative position is point-wise bounded by* μ *in the Bruhat order and* $\lambda : \mathscr{T} \to \mathscr{G}_b|_{\mathcal{Y}^{R^+}_{[r,\infty)}}$ *is an (equivalence class of) isogenies.*

Remark 2.3.3. *In definition 2.3.2, let* $E(\mu)$ *denote the reflex field of* μ*. Since* $\mathscr{G}$ *is reductive over* $\mathbb{Z}_p$*,* $E(\mu)$ *is an unramified extension of* $\mathbb{Q}_p$*. Moreover, since* $\mathfrak{f}$ *comes equipped with an inclusion* $\overline{\mathbb{F}}_p \to \mathfrak{f}$ *we get an inclusion* $E(\mu) \to W(\mathfrak{f})[\frac{1}{p}]$*. We are implicitly using this inclusion of fields to compare the relative positions.*

The purpose of this section is to prove that the $\mathrm{Sht}^{\mathscr{G}_b, \leq \mu}_{W(\mathfrak{f})}$ are rich p-smelted kimberlites that have connected p-adic tubular neighborhoods.

2.3.1 Moduli spaces of shtukas are kimberlites

In this subsection we verify that the map $\mathrm{Sht}^{\mathscr{G}_b, \leq \mu}_{W(\mathfrak{f})} \to W(\mathfrak{f})^\diamond$ forms a p-smelted kimberlite. We will need to define auxiliary spaces to simplify some of the arguments below:

Definition 2.3.4. *We let* $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ *denote the functor* $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} : \mathrm{Perf} \to \mathrm{Sets}$*:*

$$\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}(R, R^+) = \{(R^\sharp, \iota, f), M, \lambda\}$$

Where the triple $(R^\sharp, \iota, f)$ *denotes an untilt over* $\mathrm{Spa}(W(\mathfrak{f}), W(\mathfrak{f}))$, $M \in \mathscr{G}(W(R^+)[\frac{1}{\xi_{R^\sharp}}])$ *and* $\lambda : \mathscr{G}_M \to \mathscr{G}_b$ *is an equivalence class of isogenies defined over* $\mathcal{Y}^{R^+}_{[r,\infty]}$ *for some* r*. Here* $\mathscr{G}_M$ *denotes the tuple* $(\mathscr{G}, \Phi_M)$ *with* $\Phi_M : \phi^*\mathscr{G} \to \mathscr{G}$ *an isomorphism given by* M *and defined over* $\mathrm{Spec}(W(R^+)[\frac{1}{\xi_{R^\sharp}}])$.

Notice that there is a natural map $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ given by restriction and assigning $(M, \lambda) \mapsto (\mathscr{G}, \Phi_M, \lambda)$. We denote by $\mathbb{W}^+\mathscr{G}$ the sheaf in groups $\mathbb{W}^+\mathscr{G}(R, R^+) = \mathscr{G}(W(R^+))$, notice that $(\mathbb{W}^+\mathscr{G})_{O_C} = W^+(\mathscr{G} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[t])$ as in definition 2.2.14.

Proposition 2.3.5. *1. The functors* $\mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ *and* $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ *are small* v*-sheaves.*

2. The map $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ *is a* $\mathbb{W}^+\mathscr{G}$*-torsor for the* v*-topology.*

3. $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ *is formalizing and* $\mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ *is* v*-formalizing.*

Proof. To prove that it is a v-sheaf one has to prove that each of the entries descend. A standard argument using 2.1.20 repeatedly proves this. Given $N \in \mathbb{W}^+\mathscr{G}(R, R^+)$ and $(M, \lambda) \in \mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}(R, R^+)$ we let $N \cdot (M, \lambda) = (N^{-1}M\phi(N), \lambda \circ N)$. This specifies an action of $\mathbb{W}^+\mathscr{G}$ on $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ that makes the map $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ equivariant when the target is endowed with the trivial action. It is enough to prove that the basechange of $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ along product of points gives a trivial $\mathbb{W}^+\mathscr{G}$-torsor.

Let $\mathrm{Spa}(R, R^+)$ be a product of points, and take $(\mathscr{T}, \Phi, \lambda) \in \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}(R, R^+)$. We can glue $\mathscr{T}$ along λ over $\mathcal{Y}^{R^+}_{[r,\infty)}$ to extend to a $\mathscr{G}$-bundle over $\mathcal{Y}_{R^+}$, a meromorphic isomorphism Φ over $\mathcal{Y}_{R^+} \setminus V(\xi_{R^\sharp})$ and an isogeny λ over $\mathcal{Y}^{R^+}_{[r,\infty]}$. We can use theorem 2.1.12 and theorem 2.1.18 to get a $\mathscr{G}$-bundle over $\mathrm{Spec}(W(R^+))$ with a meromorphic Φ that restrict to the previous ones. Since R^+ is a product of valuation rings with algebraically closed fraction field any $\mathscr{G}$-bundle on $\mathrm{Spec}(W(R^+))$ is trivial. This is the case because $\mathrm{Spec}(W(R^+))$ splits every étale cover. The choice of a trivialization specifies a section $(M, \lambda) \in \mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}(R, R^+)$ and after chasing definitions one can see that the natural action of $\mathbb{W}^+\mathscr{G}$ on the set of trivialization acts compatibly with the action specified above.

We prove that $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ is formalizing, this already implies that $\mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ is v-formalizing since the map $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ is surjective. Let $\mathrm{Spa}(S, S^+) \in \mathrm{Perf}$, fix $\varpi_S \in S^+$ a pseudo-uniformizer and take

$$((S^\sharp, \iota, f), M, \lambda) \in \mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}(S, S^+)$$

Given a map $f : \mathrm{Spa}(L, L^+) \to \mathrm{Spd}(S^+, S^+)$ we get a map of rings $f : W(S^+)[\frac{1}{\xi_{S^\sharp}}] \to W(L^+)[\frac{1}{\xi_{L^\sharp}}]$, and we can let M_L be $f(M)$. Moreover, fix a pseudo-uniformizer $\varpi_L \in L^+$, we claim that for any such choice and for any $r \in \mathbb{R}$ there is a large enough $r' \in \mathbb{R}$ for which the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{Y}^{L^+}_{[r',\infty]} & \longrightarrow & \mathcal{Y}^{R^+}_{[r,\infty]} \\ \downarrow & & \downarrow \\ \mathrm{Spa}(W(L^+), W(L^+)) & \longrightarrow & \mathrm{Spa}(W(S^+), W(S^+)) \end{array}$$

This map allows us to pullback the isogeny λ to $\mathrm{Spa}(L, L^+)$. The equivalence class of isogenies constructed this way does not depend of the choices of ϖ_S, ϖ_L, r or r' and the construction is functorial, so it defines a map $\mathrm{Spd}(S^+, S^+) \to \mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$. $\square$

Moduli spaces of shtukas satisfy the valuative criterion for partial properness over $W(\mathfrak{f})^\diamond$ since the definition of all of the data involved (via Tannakian formalism) takes place in the exact category of vector bundles over $\mathcal{Y}^{R^+}_{[0,\infty)}$ which is equivalent (by an exact equivalence) to the category of vector bundles over $\mathcal{Y}^{R^\circ}_{[0,\infty)}$.

Lemma 2.3.6. *Let $\mathscr{G}_1 \to \mathscr{G}_2$ be a closed embeddings of reductive groups over $\mathbb{Z}_p$ and $\mathscr{G}_b$ an isocrystal with $\mathscr{G}_1$ structure. Let $\mathscr{G}'_b = \mathscr{G}_b \overset{\mathscr{G}_1}{\times} \mathscr{G}_2$, the induced map $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{LSht}^{\mathscr{G}'_b}_{W(\mathfrak{f})}$ is a closed immersion.*

Proof. It is enough to prove that the basechange by any totally disconnected perfectoid space is a closed immersion. Let $\mathrm{Spa}(S, S^+)$ in Perf be totally disconnected, and let $(M, \lambda) \in \mathrm{LSht}^{\mathscr{G}'_b}_{W(\mathfrak{f})}(S, S^+)$. Abusing notation we let λ denote a choice of representative of the equivalence class of isogenies and we let $r \in \mathbb{R}$ such that λ is defined over $\mathcal{Y}^{S^+}_{[r,\infty]}$. By unraveling the definitions we can think of M as a ring map $\mathcal{O}_{\mathscr{G}_2} \to W(S^+)[\frac{1}{\xi_{S^\sharp}}]$ and we think of λ as a ring map $\mathcal{O}_{\mathscr{G}_2} \to B^{[r,\infty]}_{S^+}$ (with the notation as in lemma 2.1.24). Since $\mathscr{G}_1 \to \mathscr{G}_2$ is a closed embedding of affine algebraic groups we have that $\mathcal{O}_{\mathscr{G}_1} = \mathcal{O}_{\mathscr{G}_2}/I$ for some finitely generated ideal $I \subseteq \mathcal{O}_{\mathscr{G}_2}$. The basechange

$$\mathrm{Spa}(S, S^+) \times_{\mathrm{LSht}^{\mathscr{G}'_b}_{W(\mathfrak{f})}} \mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$$

is representing the moduli of maps $\mathrm{Spa}(R, R^+) \to \mathrm{Spa}(S, S^+)$ for which the compositions:

$$M : \mathcal{O}_{\mathscr{G}_2} \to W(S^+)[\frac{1}{\xi_{S^\sharp}}] \to W(R^+)[\frac{1}{\xi_{R^\sharp}}]$$
$$\lambda : \mathcal{O}_{\mathscr{G}_2} \to B^{S^+}_{[r,\infty]} \to B^{R^+}_{[r,\infty]}$$

map elements of I to 0.

Let us prove that for any element $t \in W(S^+)[\frac{1}{\xi_{S^\sharp}}]$ (or $t \in B^{[r,\infty]}_{S^+}$) the moduli of points in $\mathrm{Spa}(S, S^+)$ where t is identically 0 forms a closed immersion. Fix $t \in W(S^+)[\frac{1}{\xi_{S^\sharp}}]$, since $\xi_{S^\sharp}$ is not a zero-divisor the moduli of points where t is 0 is the same as that of $\xi^n \cdot t$ so we may assume $t \in W(S^+)$. Using the Teichműller expansion we have $t \in (S^+)^{\mathbb{N}}$ and t is 0 if and only if each entry is 0. This defines a Zariski closed subset of $\mathrm{Spa}(S, S^+)$. We prove the other case, fix $t \in B^{[r,\infty]}_{S^+} \subseteq B^{[r,\infty)}_{S^+}$ and let $Z \subseteq |\mathcal{Y}^{R^+}_{[r,\infty)}|$ be the set of valuations with $|t|_z = 0$. We have a projection map of diamonds $\pi : (\mathcal{Y}^{S^+}_{[r,\infty)})^\lozenge \to \mathrm{Spd}(S, S^+)$ which is ℓ-cohomologically smooth and consequently universally open (See [51] 24.5). The moduli of points we are considering is given by maps to $\mathrm{Spa}(S, S^+)$ that factor through $Z' = |\mathrm{Spa}(S, S^+)| \setminus \pi(|\mathcal{Y}^{R^+}_{[r,\infty)}| \setminus Z)$ which is a closed subset. Since the subset of interest is closed and generalizing, it defines a closed immersion of $\mathrm{Spa}(S, S^+)$ (See [51] 7.6). □

Proposition 2.3.7. *With notation as in lemma 2.3.6 the map* $\mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}'_b}_{W(\mathfrak{f})}$ *is a closed immersion. In particular, if we let* $\mathscr{G}_2 = \mathscr{G}_1 \times_{\mathbb{Z}_p} \mathscr{G}_1$ *and we apply the result to the diagonal embedding* $\Delta : \mathscr{G}_1 \to \mathscr{G}_2$ *we deduce that* $\mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ *is separated over* $W(\mathfrak{f})^\lozenge$.

Proof. We begin by proving that the map is injective. For this consider two sets of triples

$$t_i = (\mathscr{T}_i, \Phi_i, \lambda_i) \in \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}(R, R^+) \; with \, i \in \{1, 2\}$$

and suppose that the $t_i \overset{\mathscr{G}_1}{\times} \mathscr{G}_2 := (\mathscr{T}_i \overset{\mathscr{G}_1}{\times} \mathscr{G}_2, \Phi_i, \lambda_i)$ become isomorphic, we need to prove $t_1 \cong t_2$. Since products of points form a basis for the v-topology we can assume $\mathrm{Spa}(R, R^+)$ to be a product of points. For a product of points any map $\mathrm{Spa}(R, R^+) \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ factors through $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$. Let $T_i \in \mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})}(R, R^+)$ factoring t_i with $T_i := (M_i, \lambda_i)$. The set of choices for T_i mapping to t_i forms a $\mathbb{W}^+\mathscr{G}_1(R, R^+)$-torsors. Since $t_1 \overset{\mathscr{G}_1}{\times} \mathscr{G}_2 \cong t_2 \overset{\mathscr{G}_1}{\times} \mathscr{G}_2$ we have that $T_1 \overset{\mathscr{G}_1}{\times} \mathscr{G}_2$ and $T_2 \overset{\mathscr{G}_1}{\times} \mathscr{G}_2$ are in the same $\mathbb{W}^+\mathscr{G}_2(R, R^+)$-orbit. But $\lambda_i \in \mathscr{G}_1(B^{[r,\infty)}_{R^+})$ so that $\lambda_1 \circ \lambda_2^{-1} \in \mathscr{G}_1(B^{[r,\infty)}_{R^+}) \cap \mathscr{G}_2(W(R^+))$, since $W(R^+) \to B^{[r,\infty)}_{R^+}$ is injective this intersection is $\mathscr{G}_1(W(R^+))$. This, together with the injectivity of lemma 2.3.6, proves that T_1 and T_2 are in the same $\mathbb{W}^+\mathscr{G}_1$-orbit, which proves $t_1 = t_2$.

Once we know $\mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}'_b}_{W(\mathfrak{f})}$ is injective it is enough prove that the map is proper for it to be a closed immersion. Injectivity implies the map is a separated map of v-sheaves and since each of them satisfies the valuative criterion of partial properness over $\mathbb{F}_p^\lozenge$, the map between them is a partially proper map. We only have left to prove that the map $\mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}'_b}_{W(\mathfrak{f})}$ is quasi-compact. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} & \longrightarrow & \mathrm{LSht}^{\mathscr{G}'_b}_{W(\mathfrak{f})} \\ \downarrow & & \downarrow \\ \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})} & \longrightarrow & \mathrm{Sht}^{\mathscr{G}'_b}_{W(\mathfrak{f})} \end{array}$$

The composition $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}'_b}_{W(\mathfrak{f})}$ is a quasi-compact map, and the map $\mathrm{LSht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ is surjective which formally implies that the map $\mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}'_b}_{W(\mathfrak{f})}$ is quasi-compact. □

Proposition 2.3.8. *For any* $\mu \in X_*^+(\mathfrak{T}_{\overline{\mathbb{Q}}_p})$ *we have that* $\mathrm{Sht}^{\mathscr{G}_b, \leq \mu}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ *is a closed immersion. Moreover,* $\mathrm{Sht}^{\mathscr{G}_b, \leq \mu}_{W(\mathfrak{f})}$ *is v-formalizing.*

Proof. Let $\mathrm{LSht}_{W(\mathfrak{f})}^{\mathscr{G}_b,\leq\mu}$ denote the basechange of $\mathrm{LSht}_{W(\mathfrak{f})}^{\mathscr{G}_b} \to \mathrm{Sht}_{W(\mathfrak{f})}^{\mathscr{G}_b}$ by $\mathrm{Sht}_{W(\mathfrak{f})}^{\mathscr{G}_b,\leq\mu}$. Given an element $(M,\lambda) \in \mathrm{LSht}_{W(\mathfrak{f})}^{\mathscr{G}_b}(R,R^+)$, M naturally defines an (R,R^+)-valued point of $\mathrm{Gr}_{W(\mathfrak{f})}^{\mathscr{G}}$ when we think of M as an element of $\mathscr{G}(B_{dR}(R^\sharp))$.

We have the following pair of Cartesian diagrams:

$$\begin{tikzcd}
\mathrm{LSht}_{W(\mathfrak{f})}^{\mathscr{G}_b,\leq\mu} \arrow[r] \arrow[d] & \mathrm{LSht}_{W(\mathfrak{f})}^{\mathscr{G}_b} \arrow[d] \\
\mathrm{Gr}_{W(\mathfrak{f})}^{\mathscr{G},\leq\mu} \arrow[r] & \mathrm{Gr}_{W(\mathfrak{f})}^{\mathscr{G}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{LSht}_{W(\mathfrak{f})}^{\mathscr{G}_b,\leq\mu} \arrow[r] \arrow[d] & \mathrm{LSht}_{W(\mathfrak{f})}^{\mathscr{G}_b} \arrow[d] \\
\mathrm{Sht}_{W(\mathfrak{f})}^{\mathscr{G}_b,\leq\mu} \arrow[r] & \mathrm{Sht}_{W(\mathfrak{f})}^{\mathscr{G}_b}
\end{tikzcd}$$

Since being a closed immersion can be checked v-locally on the target (See [51] 10.11), and since $\mathrm{LSht}_{W(\mathfrak{f})}^{\mathscr{G}_b} \to \mathrm{Sht}_{W(\mathfrak{f})}^{\mathscr{G}_b}$ is surjective $\mathrm{Sht}_{W(\mathfrak{f})}^{\mathscr{G}_b,\leq\mu} \to \mathrm{Sht}_{W(\mathfrak{f})}^{\mathscr{G}_b}$ is a closed immersion. Moreover, by 2.2.5 the map $\mathrm{Gr}_{W(\mathfrak{f})}^{\mathscr{G},\leq\mu} \to \mathrm{Gr}_{W(\mathfrak{f})}^{\mathscr{G}}$ is formally adic which implies that $\mathrm{LSht}_{W(\mathfrak{f})}^{\mathscr{G}_b,\leq\mu}$ is formalizing and consequently that $\mathrm{Sht}_{W(\mathfrak{f})}^{\mathscr{G}_b,\leq\mu}$ is v-formalizing. □

In what follows we will prove that the functors $(\mathrm{Sht}_{W(\mathfrak{f})}^{\mathscr{G}_b,\leq\mu})^{\mathrm{red}}$ are represented by affine Deligne Lusztig varieties. We warn the reader that the definition that we take of affine Deligne Lusztig varieties is not the standard one. Nevertheless, it is well known and easy to establish that the definition we take defines the same objects as the standard definition.

Definition 2.3.9. *Let $\mathscr{G}_b$ be an isocrystal with $\mathscr{G}$-structure and $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}} \to \mathfrak{T}_{\overline{\mathbb{Q}}_p}$ a cocharacter. We define the v-sheaf $X_{\leq\mu}^{\mathscr{G}_b} : \mathrm{PCAlg}_{/\mathfrak{f}}^{op} \to \mathrm{Sets}$ as:*

$$X_{\leq\mu}^{\mathscr{G}_b}(R) = \{(\mathscr{T},\Phi,\lambda)\}/\cong$$

Where $\mathscr{T}$ is a $\mathscr{G}$-torsor over $\mathrm{Spec}(W(R))$, $\Phi : \phi^*\mathscr{T} \to \mathscr{T}$ is an isomorphism defined over $\mathrm{Spec}(W(R)[\frac{1}{p}])$ of relative position bounded by μ and $\lambda : \mathscr{T} \to \mathscr{G}_b$ is a ϕ-equivariant isomorphism over $\mathrm{Spec}(W(R)[\frac{1}{p}])$

Proposition 2.3.10. *We have an identification $X_{\leq\mu}^{\mathscr{G}_b} = (\mathrm{Sht}_{W(\mathfrak{f})}^{\mathscr{G}_b,\leq\mu})^{\mathrm{red}}$. Moreover, the map $(X_{\leq\mu}^{\mathscr{G}_b})^\diamond \to \mathrm{Sht}_{W(\mathfrak{f})}^{\mathscr{G}_b,\leq\mu}$ is injective and $\mathrm{Sht}_{W(\mathfrak{f})}^{\mathscr{G}_b,\leq\mu}$ is a specializing v-sheaf.*

Proof. Given a map $\mathrm{Spec}(R) \to X_{\leq\mu}^{\mathscr{G}_b}$ we construct functorially a map $\mathrm{Spec}(R)^\diamond \to \mathrm{Sht}_{W(\mathfrak{f})}^{\mathscr{G}_b,\leq\mu}$ in what follows. The untilt is always the characteristic p untilt. For any perfectoid space $f : \mathrm{Spa}(S,S^+) \to \mathrm{Spec}(R)^\diamond$ we get a triple $(f^*\mathscr{T}, f^*\Phi, f^*\lambda)$ coming from the map of rings $f : W(R) \to W(S^+)$ and by restriction to the appropriate loci $\mathcal{Y}_{[0,\infty)}^{S^+}$, $\mathcal{Y}_{[0,\infty)}^{S^+} \setminus V(p)$ and $\mathcal{Y}_{[r,\infty)}^{S^+}$ respectively. This data defines functorially a map $\mathrm{Spa}(S,S^+) \to \mathrm{Sht}_{W(\mathfrak{f})}^{\mathscr{G}_b,\leq\mu}$, consequently a map $\mathrm{Spec}(R)^\diamond \to \mathrm{Sht}_{W(\mathfrak{f})}^{\mathscr{G}_b,\leq\mu}$. The construction of $\mathrm{Spec}(R)^\diamond \to \mathrm{Sht}_{W(\mathfrak{f})}^{\mathscr{G}_b,\leq\mu}$ is clearly functorial in $\mathrm{PCAlg}_{/k}^{op}$. By adjunction, this gives a map $X_{\leq\mu}^{\mathscr{G}_b} \to (\mathrm{Sht}_{W(\mathfrak{f})}^{\mathscr{G}_b,\leq\mu})^{\mathrm{red}}$, we claim this map is an isomorphism.

We begin by proving it is injective, since $X^{\mathscr{G}_b}_{\leq\mu}$ is represented by a perfect scheme

$$((X^{\mathscr{G}_b}_{\leq\mu})^{\diamondsuit})^{\mathrm{red}} = X^{\mathscr{G}_b}_{\leq\mu}$$

and we may prove that the map $(X^{\mathscr{G}_b}_{\leq\mu})^{\diamondsuit} \to \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ is injective instead. Take two arbitrary maps $g_i : \mathrm{Spa}(R, R^+) \to (X^{\mathscr{G}_b}_{\leq\mu})^{\diamondsuit}$, it is enough to prove injectivity v-locally so we may assume the maps factor through maps of the form $g_i' : \mathrm{Spec}(R^+)^{\diamondsuit} \to (X^{\mathscr{G}_b}_{\leq\mu})^{\diamondsuit}$. The g_i' are given by data $(\mathscr{T}_i, \Phi_i, \lambda_i)$ over $\mathrm{Spec}(W(R^+))$, $\mathrm{Spec}(W(R^+)[\frac{1}{p}])$ and $\mathrm{Spec}(W(R^+)[\frac{1}{p}])$ respectively and the g_i are given by restricting these data to $\mathcal{Y}^{R^+}_{[0,\infty)}$, $\mathcal{Y}^{R^+}_{[0,\infty)} \setminus V(p)$ and $\mathcal{Y}^{R^+}_{[r,\infty)}$ respectively. Nevertheless, we can recover g_i' from g_i since we can use λ_i to glue back the restricted data as in the proof of proposition 2.3.5.

Let us now prove surjectivity. Let $f : \mathrm{Spec}(A)^{\diamondsuit} \to \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ and $g : \mathrm{Spa}(R, R^+) \to \mathrm{Spec}(A)^{\diamondsuit}$ be a map with $\mathrm{Spa}(R, R^+)$ a product of points and $A \in \mathrm{PCAlg}^{op}_{/\mathfrak{f}}$. We will show below how to construct the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Spa}(R, R^+) \arrow[r] \arrow[d] & \mathrm{Spec}(A)^{\diamondsuit} \arrow[d] \\
(X^{\mathscr{G}_b}_{\leq\mu})^{\diamondsuit} \arrow[r] & \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}
\end{tikzcd}$$

Since products of points are a basis for the topology, and since $(X^{\mathscr{G}_b}_{\leq\mu})^{\diamondsuit} \to \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ is injective this defines a map $\mathrm{Spec}(A)^{\diamondsuit} \to (X^{\mathscr{G}_b}_{\leq\mu})^{\diamondsuit}$ factoring our original map to $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ and proves the desired surjectivity.

Fix a pseudo-uniformizer $\varpi \in R^+$, we let $\mathrm{Spa}(R_\infty, {R_\infty}^+)$ be a second product of points defined as follows: $R^+_\infty = \prod_{i=1}^\infty R^+$ with pseudo-uniformizer now given by $\varpi_{R_\infty} = (\varpi^i)_{i=1}^\infty$. The product of points $\mathrm{Spa}(R_\infty, {R_\infty}^+)$ comes equipped with a family of closed embeddings $\iota_i : \mathrm{Spa}(R, R^+) \to \mathrm{Spa}(R_\infty, {R_\infty}^+)$ given in coordinates by the projection onto the ith-factor. The diagonal ring map $\Delta_g : A \to \prod_{i=1}^\infty R^+$ induces a map $\Delta_g : \mathrm{Spa}(R_\infty, {R_\infty}^+) \to \mathrm{Spec}(A)^{\diamondsuit}$ with the property that $\Delta_g \circ \iota_i = g$ for every i. Since $\mathrm{Spa}(R_\infty, {R_\infty}^+)$ is a product of points, by proposition 2.1.19, the map $f \circ \Delta_g$ can be represented by a triple $(\mathcal{G}_{R_\infty}, \Phi_{R_\infty}, \lambda_{R_\infty})$ with $\mathcal{G}_{R_\infty}$ trivial. After choosing a trivialization for $\mathcal{G}_{R_\infty}$ we can think of λ_{R_∞} as a ring map $\mathcal{O}_{\mathscr{G}} \to B^{{R_\infty}^+}_{[r,\infty]}$. Moreover, since $f \circ \Delta_g \circ \iota_i = f \circ \Delta_g \circ \iota_j$ we have that for all i the composition

$$\lambda_i : \mathcal{O}_{\mathscr{G}} \to B^{{R_\infty}^+}_{[r,\infty]} \to B^{R_i^+}_{[r_i,\infty]} = B^{R^+}_{[r_i,\infty]}$$

lies in the same $\mathscr{G}(W(R^+))$-orbit. Clearly $\mathscr{G}(W({R_\infty}^+)) = \prod_{i=1}^\infty \mathscr{G}(W(R^+))$ so after a change of trivialization we may assume that $r_i = r_j$ and that $\lambda_i = \lambda_j =: \lambda_R$ for all $1 \leq i, j < \infty$. We claim that λ_R factors through the inclusion of rings $W(R^+)[\frac{1}{p}] \subseteq B^{R^+}_{[r,\infty]}$. Take an element $t \in \mathcal{O}_{\mathscr{G}}$ and consider $s = \lambda_{R_\infty}(t) \in B^{{R_\infty}^+}_{[r,\infty]}$, after replacing r by a larger number if necessary we may assume $r = n \in \mathbb{N}$. In particular, $p^k \cdot s$ lies in the p-adic completion of $W({R_\infty}^+)[\frac{[\varpi_{R_\infty}]}{p^n}]$ for some large enough $k \in \mathbb{N}$. Let us write $p^k \cdot s$ as $\sum_{j=0}^\infty x^{n(j)}[\alpha_j]p^j$ where x denotes $\frac{[\varpi_{R_\infty}]}{p^n}$,

$0 \leq n(j)$ is a multiplicity, and $\alpha_j \in R^+_\infty$. We have that $\iota_i(p^k \cdot s) = \sum_{j=0}^{\infty}(\frac{[\varpi]^i}{p^n})^{n(j)}[\iota_i(\alpha_j)]p^j$ with $\iota_i(\alpha_j) \in R^+$. In particular,

$$p^k \cdot \lambda_R(t) \in \bigcap_{i \in \mathbb{N}} (H^0(\mathcal{Y}^{R^+}_{[\frac{n}{i},\infty]}, \mathcal{O}^+)),$$

but this intersection is $W(R^+)$ proving the claim.

Since the elements of the triple $(\mathcal{G}_{R_\infty}, \Phi_{R_\infty}, \lambda_{R_\infty})$ are defined over $\mathrm{Spec}(W(R^+))$ and $\mathrm{Spec}(W(R^+)[\frac{1}{p}])$ they define a map to $\mathrm{Spec}(R^+) \to X^{\mathscr{G}_b}_{\leq \mu}$. The composition $\mathrm{Spa}(R, R^+) \to \mathrm{Spec}(R^+)^\diamondsuit \to (X^{\mathscr{G}_b}_{\leq \mu})^\diamondsuit$ gives the factorization we were looking for.

That $\mathrm{Sht}^{\mathscr{G}_b, \leq \mu}_{W(\mathfrak{f})}$ is formally separated follows from lemma 1.3.32 and proposition 2.3.7, that $\mathrm{Sht}^{\mathscr{G}_b, \leq \mu}_{W(\mathfrak{f})}$ is v-formalizing follows from proposition 2.3.8.

$\square$

Lemma 2.3.11. *The adjunction map $(X^{\mathscr{G}_b}_{\leq \mu})^\diamondsuit \to \mathrm{Sht}^{\mathscr{G}_b, \leq \mu}_{W(\mathfrak{f})}$ arising from the identification of proposition 2.3.10 is a closed immersion.*

Proof. We will use that $X^{\mathscr{G}_b}_{\leq \mu}$ admits a closed immersion into the Witt vector Grassmanian $Gr^{\mathscr{G},}_{\mathcal{W},\mathfrak{f}}$. We have that

$$(X^{\mathscr{G}_b}_{\leq \mu})^\diamondsuit = \bigcup_{\nu \in X_*(\mathfrak{T}_{\overline{\mathbb{Q}}_p})} (X^{\mathscr{G}_b}_{\leq \mu} \cap Gr^{\mathscr{G}, \leq \nu}_{\mathcal{W},\mathfrak{f}})^\diamondsuit$$

and since each of this subsheaves are coming from a perfectly finitely presented proper scheme over $\mathfrak{f}$, they are proper as v-sheaves over $\mathfrak{f}^\diamondsuit$. Consequently, the map $(X^{\mathscr{G}_b}_{\leq \mu} \cap Gr^{\mathscr{G}, \leq \nu}_{\mathcal{W},\mathfrak{f}})^\diamondsuit \to \mathrm{Sht}^{\mathscr{G}_b, \leq \mu}_{W(\mathfrak{f})}$ is proper and since it is injective a closed immersion.

Now, $X^{\mathscr{G}_b}_{\leq \mu}$ is a scheme which is locally perfectly of finite type (See [18] theorem 1.1), and in particular each point admits an open neighborhood that is spectral and Noetherian as a topological space. Using a compactness argument in the patch topology, to every point $x \in |X^{\mathscr{G}_b}_{\leq \mu}|$ we may associate an open neighborhood $U_x \subseteq X^{\mathscr{G}_b}_{\leq \mu}$ and finite number of $\nu_i \in X^+_*(\mathfrak{T}_{\overline{\mathbb{Q}}_p})$ for which $U = U \cap (\bigcup_{i \in I_x} Gr^{\mathscr{G}, \leq \nu_i}_{\mathcal{W},\mathfrak{f}})$. Indeed, if U_x is Noetherian every closed subset is open in the constructible topology.

In proposition 2.3.10, we proved that $\mathrm{Sht}^{\mathscr{G}_b, \leq \mu}_{W(\mathfrak{f})}$ is a specializing v-sheaf, so by proposition 1.4.15 we get a specialization map $\mathrm{sp}_{\mathrm{Sht}^{\mathscr{G}_b, \leq \mu}_{W(\mathfrak{f})}} : |\mathrm{Sht}^{\mathscr{G}_b, \leq \mu}_{W(\mathfrak{f})}| \to |X^{\mathscr{G}_b}_{\leq \mu}|$. We let $V_x = (\mathrm{sp}_{\mathrm{Sht}^{\mathscr{G}_b, \leq \mu}_{W(\mathfrak{f})}})^{-1}(U_x)$ for $x \in |X^{\mathscr{G}_b}_{\leq \mu}|$ and U_x as above, this forms an open cover of $\mathrm{Sht}^{\mathscr{G}_b, \leq \mu}_{W(\mathfrak{f})}$. Since being a closed immersion is v-local on the target and $V_x \to \mathrm{Sht}^{\mathscr{G}_b, \leq \mu}_{W(\mathfrak{f})}$ is a formally adic open immersion it is enough to verify that $(V_x^{\mathrm{red}})^\diamondsuit \to V_x$ is a closed immersion. But the adjunction map $(U_x)^\diamondsuit \to V_x$ fits in the following Cartesian diagram:

$$\begin{tikzcd}
U_x^\lozenge \arrow[r, "Id"] \arrow[d] & U_x^\lozenge \arrow[r] \arrow[d] & V_x \arrow[d] \\
(\bigcup_{i\in I_x} Gr^{\mathscr{G},\leq \nu_i}_{\mathcal{W},\mathfrak{f}} \cap X^{\mathscr{G}_b}_{\leq \mu})^\lozenge \arrow[r] & (X^{\mathscr{G}_b}_{\leq\mu})^\lozenge \arrow[r] & \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}
\end{tikzcd}$$

Since the union of a finite number of closed immersions still defines a closed immersion we can conclude by basechange that $U_x^\lozenge \to V_x$ is also a closed immersion.

$\square$

Proposition 2.3.12. *With the notation as above the map $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})} \to W(\mathfrak{f})^\lozenge$ is a p-smelted kimberlite as in definition 1.4.18.*

Proof. Proposition 2.3.10 proves that $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ is a specializing v-sheaf, in proposition 2.3.10 we proved that $(\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})})^{\mathrm{red}}$ is represented by a scheme and by lemma 2.3.11 the adjunction map $((\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})})^{\mathrm{red}})^\lozenge \to \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ is a closed immersion which finished the proof that $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$ is prekimberlite. Theorem 23.1.4 of [53] proves that $(\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})})_\eta$ is a locally spatial diamond. $\square$

2.3.2 Comparison of tubular neighborhoods

Recall that in this section $\mathscr{G}$ is a reductive group over $\mathrm{Spec}(\mathbb{Z}_p)$, let $\mathcal{D} = (\mathcal{D}, \Phi_\mathcal{D})$ be a $\mathscr{G}$-torsor over $\mathrm{Spec}(W(\mathfrak{f}))$ together with an isomorphism $\Phi_\mathcal{D} : \phi^*\mathcal{D} \to \mathcal{D}$ defined over $\mathrm{Spec}(W(\mathfrak{f})[\frac{1}{p}])$, and fix $\mu \in X_*^+(\mathfrak{T}_{\overline{\mathbb{Q}}_p})$. We can define some objects associated to this data, which are nothing but "coordinate-free" versions of the moduli we defined in the previous sections:

Definition 2.3.13. 1. *We denote the functor $\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})} : \mathrm{Perf}_\mathfrak{f} \to \mathrm{Sets}$ with*

$$\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}(R, R^+) = \{((R^\sharp, \iota, f), \mathscr{T}, \psi)\}/\cong$$

Where $(R^\sharp, \iota, f)$ is an untilt over $W(\mathfrak{f})$, $\mathscr{T}$ is a $\mathscr{G}$-torsor over $\mathcal{Y}_{R^+}$ and $\psi : \mathscr{T} \to \mathcal{D}$ is an isomorphism defined over $\mathcal{Y}_{R^+} \setminus V(\xi_{R^\sharp})$ that is meromorphic along $\xi_{R^\sharp}$.

2. *We denote the functor $\mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})} : \mathrm{Perf}_\mathfrak{f} \to \mathrm{Sets}$ with*

$$\mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}(R, R^+) = \{((R^\sharp, \iota, f), \mathscr{T}, \Phi, \lambda)\}/\cong$$

Where $(R^\sharp, \iota, f)$ is an untilt over $W(\mathfrak{f})$, $(\mathscr{T}, \Phi)$ is a shtuka with $\mathscr{G}$-structure, and $\lambda : \mathscr{T} \to \mathcal{D}$ is an isogeny.

The functors $\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}$ and $\mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}$ come with a canonical section $can_\mathcal{D} : \mathrm{Spec}(\mathfrak{f})^\lozenge \to \mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}$ given by the data $(\phi^*\mathcal{D}, \Phi_\mathcal{D})$ and $can_\mathcal{D} : \mathrm{Spec}(\mathfrak{f})^\lozenge \to \mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}$ given by $(\mathcal{D}, \Phi_\mathcal{D}, Id)$ respectively. We point out that if we fix an isomorphism $\tau : \mathcal{D} \to \mathscr{G}$ we get isomorphisms $\tau : \mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})} \to \mathrm{Gr}^{\mathscr{G}}_{W(\mathfrak{f})}$, and $\tau : \mathrm{Gr}^{\mathcal{D},\leq\mu}_{W(\mathfrak{f})} \to \mathrm{Gr}^{\mathscr{G},\leq\mu}_{W(\mathfrak{f})}$. Moreover, if we are given a section $\sigma : \mathrm{Spec}(\mathfrak{f})^\lozenge \to \mathrm{Gr}^{\mathscr{G}}_{W(\mathfrak{f})}$ we can construct a pair $(\mathcal{D}, \Phi_\mathcal{D})$ and an isomorphism $\tau : \mathcal{D} \to \mathscr{G}$ such that the following diagram is commutative:

$$\begin{tikzcd}
 & \mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})} \arrow[dd, "\tau"] \\
\mathrm{Spec}(\mathfrak{f})^{\diamondsuit} \arrow[ur, "can_{\mathcal{D}}"] \arrow[dr, "\sigma"'] & \\
 & \mathrm{Gr}^{\mathscr{G}}_{W(\mathfrak{f})}
\end{tikzcd}$$

Analogously, if we find a ϕ-equivariant isomorphism $\tau : \mathcal{D} \to \mathscr{G}_b$ over $\mathrm{Spec}(W(\mathfrak{f})[\frac{1}{p}])$ we get an isomorphism $\tau : \mathrm{Sht}^{\mathcal{D},\leq\mu}_{W(\mathfrak{f})} \to \mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})}$, and given a section $\sigma : \mathrm{Spec}(\mathfrak{f})^{\diamondsuit} \to \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}$ we can construct $(\mathcal{D}, \Phi_{\mathcal{D}})$ and τ making the following diagram commutative:

$$\begin{tikzcd}
 & \mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})} \arrow[dd, "\tau"] \\
\mathrm{Spec}(\mathfrak{f})^{\diamondsuit} \arrow[ur, "can_{\mathcal{D}}"] \arrow[dr, "\sigma"'] & \\
 & \mathrm{Sht}^{\mathscr{G}_b}_{W(\mathfrak{f})}
\end{tikzcd}$$

Since $\mathfrak{f}$ is algebraically closed every tubular neighborhood of Grassmanians and moduli of mixed characteristic shtukas at closed points are coming from the canonical one associated to some pair $(\mathcal{D}, \Phi_{\mathcal{D}})$. Indeed, every closed point of $X^{\mathscr{G}_b}_{\leq\mu}$ and $\mathrm{Gr}^{\mathscr{G}}_{\mathcal{W},\mathfrak{f}}$ is the image of a section since the bounded version of these ind-schemes are locally perfectly of finite presentation over $\mathrm{Spec}(\mathfrak{f})$.

Theorem 2.3.14. *Given $(\mathcal{D}, \Phi_{\mathcal{D}})$ and $\mu \in X_*(\mathfrak{T}_{\overline{\mathbb{Q}}_p})$ as above, and with notation as below we have a local model diagram:*

$$\begin{tikzcd}
 & \widehat{\mathrm{LSht}^{\leq\mu}_{\mathcal{D}}} = \widehat{LGr^{\leq\mu}_{\mathcal{D}}} \arrow[dl] \arrow[dr] & \\
\widehat{\mathrm{Sht}^{\mathcal{D},\leq\mu}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}} & & \widehat{\mathrm{Gr}^{\mathcal{D},\leq\mu}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}
\end{tikzcd}$$

Moreover, both arrows are $\widehat{LG}_{\mathcal{D}}$-torsors. In particular, $\widehat{\mathrm{Sht}^{\mathcal{D},\leq\mu}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}$ is non-empty and connected if and only if $\widehat{\mathrm{Gr}^{\mathcal{D},\leq\mu}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}$ is.

Before proving the theorem we will need some preparation.

Definition 2.3.15. *1. We let $\widehat{LG}_{\mathcal{D}}$ denote the sheaf of groups over $W(\mathfrak{f})^{\diamondsuit}$ given by*

$$\widehat{LG}_{\mathcal{D}}(R, R^+) = \{((R^\sharp, \iota, f), g)\}.$$

Where $(R^\sharp, \iota, f)$ is an untilt over $W(\mathfrak{f})$ and $g : \mathcal{D} \to \mathcal{D}$ is an automorphism of $\mathscr{G}$-torsors defined over $\mathrm{Spec}(W(R^+))$ *for which there is a pseudo-uniformizer $\varpi_g \in R^+$, depending of g, such that g restricts to the identity over* $\mathrm{Spec}(W(R^+)/[\varpi_g])$. *We define $\widehat{LG}_{\phi^*\mathcal{D}}$ in a similar way exchanging the role of $\mathcal{D}$ for that of $\phi^*\mathcal{D}$.*

2. *We let $\widehat{LGr}_{\mathcal{D}}$ be the v-sheaf over $W(\mathfrak{f})^\diamond$ assigning*

$$\widehat{LGr}_{\mathcal{D}}(R, R^+) = \{(R^\sharp, \iota, f), \mathscr{T}, \psi, \sigma\}/\cong$$

Where $(R^\sharp, \iota, f)$ is an untilt over $W(\mathfrak{f})$, $\mathscr{T}$ is a $\mathscr{G}$-torsor over $\mathrm{Spec}(W(R^+))$, *$\psi : \mathscr{T} \to \mathcal{D}$ is an isomorphism over* $\mathrm{Spec}(W(R^+)[\frac{1}{\xi}])$ *and $\sigma : \mathscr{T} \to \phi^*\mathcal{D}$ is an isomorphism of $\mathscr{G}$-torsors over* $\mathrm{Spec}(W(R^+))$ *such that there is a pseudo-uniformizer $\varpi \in R^+$ depending on the data for which $\Phi_{\mathcal{D}} \circ \sigma = \psi$ when restricted to* $\mathrm{Spec}(W(R^+)/[\varpi])$. *We may also add a boundedness condition on ψ to obtain $\widehat{LGr^{\leq\mu}_{\mathcal{D}}}$.*

3. *We let $\widehat{\mathrm{LSht}}_{\mathcal{D}}$ be the v-sheaf over $W(\mathfrak{f})^\diamond$ assigning*

$$\widehat{\mathrm{LSht}}_{\mathcal{D}}(R, R^+) = \{(R^\sharp, \iota, f), \mathscr{T}, \Phi, \lambda, \sigma\}/\cong$$

Where $(R^\sharp, \iota, f)$ is an untilt over $W(\mathfrak{f})$, $\mathscr{T}$ is a $\mathscr{G}$-torsor over $\mathrm{Spec}(W(R^+))$, *$\Phi : \phi^*\mathscr{T} \to \mathscr{T}$ is an isomorphism over* $\mathrm{Spec}(W(R^+)[\frac{1}{\xi}])$, *$\lambda : \mathscr{T} \to \mathcal{D}$ is an isogeny over $\mathcal{Y}^{R^+}_{[r,\infty]}$ and $\sigma : \mathscr{T} \to \mathcal{D}$ is an isomorphism of $\mathscr{G}$-torsors over* $\mathrm{Spec}(W(R^+))$ *such that there is a pseudo-uniformizer $\varpi \in R^+$ depending on the data for which $\sigma = \lambda$ when restricted to* $\mathrm{Spec}(B^{R^+}_{[r,\infty]}/[\varpi])$. *We may also add a boundedness condition on Φ to obtain $\widehat{\mathrm{LSht}^{\leq\mu}_{\mathcal{D}}}$.*

Standard arguments using proposition 2.1.20 will prove that the objects in definition 2.3.15 are v-sheaves. Notice though, that the category of vector bundles over $\mathrm{Spec}(W(R^+))$ fibered over Perf does not form a stack for the v-topology. Nevertheless, the category fibered over Perf that assigns to $\mathrm{Spa}(R, R^+)$ the category of pairs $(\mathscr{T}, \sigma)$ where $\mathscr{T}$ is a $\mathscr{G}$-torsor over $\mathrm{Spec}(W(R^+))$ and $\sigma : \mathscr{T} \to \mathscr{G}$ is a trivialization does form a stack for the v-topology on Perf.

There is a natural map $\widehat{LGr}_{\mathcal{D}} \to \mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}$ that takes a triple $(\mathscr{T}, \psi, \sigma)$ and assigns the pair $(\mathscr{T}, \psi)$ restricted to $\mathcal{Y}^{R^+}_{[0,\infty)}$ and $\mathcal{Y}^{R^+}_{[0,\infty)} \setminus V(\xi)$ respectively. This map is $\widehat{LG}_{\phi^*\mathcal{D}}$-equivariant when we consider the left action $\widehat{LG}_{\phi^*\mathcal{D}} \times \widehat{LGr}_{\mathcal{D}} \to \widehat{LGr}_{\mathcal{D}}$ sending an element $(g, (\mathscr{T}, \psi, \sigma))$ to $(\mathscr{T}, \psi, g \circ \sigma)$ and $\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}$ is given the trivial $\widehat{LG}_{\phi^*\mathcal{D}}$-action.

Lemma 2.3.16. *The natural map $\widehat{LGr}_{\mathcal{D}} \to \mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}$ factors through $\widehat{\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}$. Moreover, the map $\widehat{LGr}_{\mathcal{D}} \to \widehat{\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}$ is a $\widehat{LG}_{\phi^*\mathcal{D}}$-torsor.*

Proof. We begin by proving that $\widehat{LGr}_{\mathcal{D}}$ formalizes any map coming from an affinoid perfectoid space $\mathrm{Spa}(A, A^+)$. Indeed, take a map $\mathrm{Spa}(A, A^+) \to \widehat{LGr}_{\mathcal{D}}$ given by an untilt and a triple $(\mathscr{T}, \psi, \sigma)$ and take a map $f : \mathrm{Spa}(B, B^+) \to \mathrm{Spd}(A^+, A^+)$, we have to define functorially a map $\mathrm{Spa}(B, B^+) \to \widehat{LGr_{\mathcal{D}}}$. We can construct an untilt for B functorially as in lemma 1.4.8. We have a map of affine schemes $f : \mathrm{Spec}(W(B^+)) \to \mathrm{Spec}(W(A^+))$ along which we can pullback to get the triple $(f^*\mathscr{T}, f^*\psi, f^*\sigma)$ where in this case $f^*\mathscr{T}$ is a $\mathscr{G}$-torsor over $\mathrm{Spec}(W(B^+))$, $f^*\psi : f^*\mathscr{T} \to \mathcal{D}$ is an isomorphism over $\mathrm{Spec}(W(B^+)[\frac{1}{f(\xi)}]$ and $f^*\sigma : f^*\mathscr{T} \to \phi^*\mathcal{D}$ is an isomorphism over $\mathrm{Spec}(W(B^+))$. We need to verify that this triple satisfies the constraints. Take a pseudo-uniformizer $\varpi_A \in A^+$ for which $\Phi_{\mathcal{D}} \circ \sigma = \psi$ in $\mathrm{Spec}(W(A^+)/[\varpi_A])$. By continuity, $f(\varpi_A)$ is topologically nilpotent and there is a pseudo-uniformizer $\varpi_B \in B^+$ with $f(\varpi_A) = \varpi_B \cdot t$ for some $t \in B^+$. We have $\Phi_{\mathcal{D}} \circ f^*\sigma = f^*\psi$ over $\mathrm{Spec}(W(B^+)/[\varpi_B])$ proving the constraint holds.

To prove that $\widehat{LGr_{\mathcal{D}}} \to \mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}$ factors through $\widehat{\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}$ it is enough to prove that for any map $\mathrm{Spd}(R^+, R^+) \to \widehat{LGr_{\mathcal{D}}}$ the map of reductions $(\mathrm{Spd}(R^+, R^+))^{\mathrm{red}} = \mathrm{Spec}(R^+_{\mathrm{red}})^{\diamond} \to (\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})})^{\mathrm{red}}$ factors through the canonical map $can_{\mathcal{D}} : \mathrm{Spec}(\mathfrak{f})^{\diamond} \to (\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})})^{\mathrm{red}}$. After restricting the data $(\mathscr{T}, \psi, \sigma)$ to $\mathrm{Spec}(W(R^+_{\mathrm{red}}))$ we get the identity $\Phi_{\mathcal{D}} \circ \sigma = \psi$. After pullback, the map $\mathrm{Spec}(R^+_{\mathrm{red}})^{\diamond} \to \mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}$ is given by the tuple $(\mathscr{T}, \psi)$. Since this data is isomorphic via σ to $(\phi^*\mathcal{D}, \Phi_{\mathcal{D}})$, the map factors through $can_{\mathcal{D}} : \mathrm{Spec}(\mathfrak{f})^{\diamond} \to \mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}$.

We now prove that $\widehat{LGr_{\mathcal{D}}} \to \widehat{\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}$ is surjective. It is enough to prove this for a product of points which we denote $\mathrm{Spa}(R, R^+)$, with pseudo-uniformizer $\varpi \in R^+$. In this case, by proposition 2.1.19, a (R, R^+)-valued point is given by $(\mathscr{T}, \psi)$ with $\mathscr{T}$ defined over $\mathrm{Spec}(W(R^+))$ and $\psi : \mathscr{T} \to \mathcal{D}$ defined over $\mathrm{Spec}(W(R^+)[\frac{1}{\xi}])$, with the additional condition that $(\mathscr{T}, \psi)$ is isomorphic to $(\phi^*\mathcal{D}, \Phi_{\mathcal{D}})$ when restricted to $W(R^+_{\mathrm{red}})$ and $W(R^+_{\mathrm{red}})[\frac{1}{p}]$. Such an isomorphism $\sigma_{\mathrm{red}} : (\mathscr{T}, \psi) \to (\phi^*\mathcal{D}, \Phi_{\mathcal{D}})$ is unique and has to be given by $\sigma_{\mathrm{red}} = \Phi_{\mathcal{D}}^{-1} \circ \psi_{\mathrm{red}}$, since it has to satisfy the commutative diagram:

$$\begin{tikzcd}
\mathscr{T} \arrow[dd, "\sigma_{\mathrm{red}}"'] \arrow[dr, "\psi"] & \\
& \mathcal{D} \\
\phi^*\mathcal{D} \arrow[ur, "\Phi_{\mathcal{D}}"] &
\end{tikzcd}$$

The morphism $\widetilde{\sigma} = \Phi_{\mathcal{D}}^{-1} \circ \psi : \mathscr{T} \to \phi^* D$ is defined over $\mathcal{Y}^{R^+}_{[r,\infty]}$ for r sufficiently big (so that it avoids $V(\xi)$) and it restricts to σ_{red}. We can use lemma 2.1.25 to construct an isomorphism $\sigma : \mathscr{T} \to \phi^* D$ such that $\sigma = \widetilde{\sigma}$ when restricted to $\mathrm{Spec}(B^{R^+}_{[r,\infty]}/[\varpi'])$ for some pseudo-uniformizer $\varpi' \in R^+$. In particular $\Phi_{\mathcal{D}} \circ \sigma = \psi$ over $\mathrm{Spec}(W(R^+)/[\varpi'])$. The data $(\mathscr{T}, \psi, \sigma)$ constructs a map $\mathrm{Spa}(R, R^+) \to \widehat{LGr_{\mathcal{D}}}$ which evidently composes to the original map $\mathrm{Spa}(R, R^+) \to \widehat{\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}$.

Finally, we need to prove $\widehat{LGr_{\mathcal{D}}} \times_{\mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}} \widehat{LGr_{\mathcal{D}}} \cong \widehat{LG}_{\phi^*\mathcal{D}} \times_{W(\mathfrak{f})^{\diamond}} \widehat{LGr_{\mathcal{D}}}$. Take two sets of data $(\mathscr{T}_i, \psi_i, \sigma_i)$ over $\mathrm{Spa}(A, A^+)$ and suppose that

$$(\mathscr{T}_1|_{\mathcal{Y}_{A^+}}, \psi_1|_{\mathcal{Y}_{A^+}\setminus V(\xi)}) \cong (\mathscr{T}_2|_{\mathcal{Y}_{A^+}}, \psi_2|_{\mathcal{Y}_{A^+}\setminus V(\xi)}).$$

The isomorphism must be given by $\psi_1^{-1} \circ \psi_2 : \mathscr{T}_2 \to \mathscr{T}_1$ and by the fully-faithfulness part of theorem 2.1.15 the isomorphism extends over $\mathrm{Spec}(W(A^+))$. Moreover, we can define $g = \sigma_1 \circ \psi_1^{-1} \circ \psi_2 \circ \sigma_2^{-1} : \phi^*\mathcal{D} \to \phi^*\mathcal{D}$. By hypothesis, $\sigma_1 \circ \psi_1^{-1} = \Phi_{\mathcal{D}}^{-1}$ and $\psi_2 \circ \sigma_2^{-1} = \Phi_{\mathcal{D}}$ on $\mathrm{Spec}(W(A^+)/[\varpi_A]$ for some suitable choice of pseudo-uniformizer $\varpi_A \in A^+$. Consequently we can associate to the original data the tuple $(g, \mathscr{T}_2, \psi_2, \sigma_2) \in \widehat{LG}_{\phi^*\mathcal{D}} \times_{W(\mathfrak{f})^\diamond} \widehat{LGr}_{\mathcal{D}}(A, A^+)$. On the other hand, to a tuple $(g, \mathscr{T}, \psi, \sigma)$ the action of $\widehat{LG}_{\phi^*\mathcal{D}}$ associates the pair of tuples $(\mathscr{T}, \psi, g \circ \sigma)$ and $(\mathscr{T}, \psi, \sigma)$. Since these two constructions are functorial and compose to the identity they define isomorphisms. $\square$

For moduli spaces of shtukas we have a very similar story. We have a projection map $\pi : \widehat{\mathrm{LSht}}_{\mathcal{D}} \to \mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}$, which we can construct by assigning to a tuple $(\mathscr{T}, \Phi, \lambda, \sigma)$ the tuple $(\mathscr{T}|_{\mathcal{Y}^{R^+}_{[0,\infty)}}, \Phi|_{\mathcal{Y}^{R^+}_{[0,\infty)}\setminus V(\xi)}, \lambda)$. Moreover, this projection is $\widehat{LG}_{\mathcal{D}}$-equivariant when we endow $\widehat{\mathrm{LSht}}_{\mathcal{D}}$ with the left action $\widehat{LG}_{\mathcal{D}} \times \widehat{\mathrm{LSht}}_{\mathcal{D}} \to \widehat{\mathrm{LSht}}_{\mathcal{D}}$ sending the tuple $(g, (\mathscr{T}, \Phi, \lambda, \sigma))$ to the tuple $(\mathscr{T}, \Phi, \lambda, g \circ \sigma)$ and when $\mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}$ is given the trivial action.

Lemma 2.3.17. *The natural map $\widehat{\mathrm{LSht}}_{\mathcal{D}} \to \mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}$ factors through $\widehat{\mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}$. Moreover, the map $\widehat{\mathrm{LSht}}_{\mathcal{D}} \to \widehat{\mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}$ is a $\widehat{LG}_{\mathcal{D}}$-torsor.*

Proof. The proves that $\widehat{\mathrm{LSht}}_{\mathcal{D}}$ formalizes any map $\mathrm{Spa}(A, A^+) \to \widehat{\mathrm{LSht}}_{\mathcal{D}}$ with $\mathrm{Spa}(A, A^+) \in \mathrm{Perf}$, that the map $\widehat{\mathrm{LSht}}_{\mathcal{D}} \to \mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}$ factors through $\widehat{\mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}$ and that this later map is surjective in that locus follow very similar arguments to those given in the proof of lemma 2.3.16. We omit the details.

Let us prove that $\widehat{\mathrm{LSht}}_{\mathcal{D}} \times_{\mathrm{Sht}^{\mathcal{D}}_{W(\mathfrak{f})}} \widehat{\mathrm{LSht}}_{\mathcal{D}} \cong \widehat{LG}_{\mathcal{D}} \times_{W(\mathfrak{f})^\diamond} \widehat{\mathrm{LSht}}_{\mathcal{D}}$. Take two sets of data $(\mathscr{T}_i, \Phi_i, \lambda_i, \sigma_i)$ over $\mathrm{Spa}(A, A^+)$ and suppose that

$$(\mathscr{T}_1|_{\mathcal{Y}^{A^+}_{[0,\infty)}}, \Phi_1|_{\mathcal{Y}^{A^+}_{[0,\infty)}\setminus V(\xi)}, \lambda_1) \cong (\mathscr{T}_2|_{\mathcal{Y}^{A^+}_{[0,\infty)}}, \Phi_2|_{\mathcal{Y}^{A^+}_{[0,\infty)}\setminus V(\xi)}, \lambda_2).$$

The isomorphism must be the unique lift of $\lambda_1^{-1} \circ \lambda_2 : \mathscr{T}_2 \to \mathscr{T}_1$ to $\mathcal{Y}^{A^+}_{[0,\infty)}$. Glueing along the λ_i we can also define a lift to $\mathcal{Y}_{A^+}$. Since the $\mathscr{T}_i$ are defined over $\mathrm{Spec}(W(A^+))$ and by the fully-faithfulness part of theorem 2.1.15 the isomorphism extends to $\mathrm{Spec}(W(A^+)$. Moreover, we can define $g = \sigma_1 \circ \lambda_1^{-1} \circ \lambda_2 \circ \sigma_2^{-1} : \mathcal{D} \to \mathcal{D}$. By hypothesis, $\sigma_1 \circ \lambda_1^{-1} = Id = \lambda_2 \circ \sigma_2^{-1}$ over $\mathrm{Spec}(B^{A^+}_{[r,\infty]}/[\varpi_A])$ for some suitable choice of pseudo-uniformizer $\varpi_A \in A^+$. Consequently we can associate to the original data the tuple $(g, \mathscr{T}_2, \Phi_2, \lambda_2, \sigma_2) \in \widehat{LG}_{\mathcal{D}} \times_{W(\mathfrak{f})^\diamond} \widehat{\mathrm{LSht}}_{\mathcal{D}}(A, A^+)$. The action map gives back isomorphic data. $\square$

We can now prove the theorem.

Proof. (of theorem 2.3.14). For this proof we define φ to be the inverse of Frobenious, $\varphi = \phi^{-1}$. We begin by observing there is an isomorphism $\theta : \widehat{LG}_{\mathcal{D}} \to \widehat{LG}_{\phi^*\mathcal{D}}$ given by sending $g \in \widehat{LG}_{\mathcal{D}}(R, R^+)$ with $g : \mathcal{D} \to \mathcal{D}$ to $\phi^* g : \phi^*\mathcal{D} \to \phi^*\mathcal{D}$. By definition of $\widehat{LG}_{\mathcal{D}}$, there is a pseudo-uniformizer $\varpi_g \in R^+$ for which $g = Id$ in $\mathrm{Spec}(W(R^+)/[\varpi_g])$. One can verify that $\phi^* g = Id$ in $\mathrm{Spec}(W(R^+)/[\varpi^p])$ so that $\phi^* g \in \widehat{LG}_{\phi^*\mathcal{D}}$ the inverse of this group homomorphism is of course given by sending $h \in \widehat{LG}_{\phi^*\mathcal{D}}(R, R^+)$ to $\varphi^* h$. Using θ one can then endow $\widehat{LGr}_{\mathcal{D}}$ with a $\widehat{LG}_{\mathcal{D}}$ action for which the projection $\pi : \widehat{LGr}_{\mathcal{D}} \to \mathrm{Gr}^{\mathcal{D}}_{W(\mathfrak{f})}$ of lemma 2.3.16 is a $\widehat{LG}_{\mathcal{D}}$-torsor.

In what follows we construct an isomorphism $\widehat{LGr}_{\mathcal{D}} \to \widehat{\mathrm{LSht}}_{\mathcal{D}}$. Take a perfectoid Huber pair (A, A^+) and tuple $(\mathscr{T}, \psi, \sigma) \in \widehat{LGr}_{\mathcal{D}}(A, A^+)$. Consider the $\mathscr{G}$-torsor $\varphi^*\mathscr{T}$, and consider the map $\Phi : \mathscr{T} \to \varphi^*\mathscr{T}$ defined by $\Phi = (\varphi^*\sigma)^{-1} \circ \psi$. We now construct a ϕ-equivariant map $\lambda : \varphi^*\mathscr{T} \to \mathcal{D}$. Consider the following (non-commutative!!!) diagram:

$$\begin{array}{ccc} \mathscr{T} & \xrightarrow{\sigma} & \phi^*\mathcal{D} \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi_{\mathcal{D}}} \\ \varphi^*\mathscr{T} & \xrightarrow{\varphi^*\sigma} & \mathcal{D} \end{array}$$

Each of the arrows of the diagram is defined over $\mathcal{Y}^{A^+}_{[r,\infty]}$ for big enough r avoiding $V(\xi)$. Moreover, by hypothesis there is a pseudo-uniformizer $\varpi \in A^+$ for which $\psi = \Phi_{\mathcal{D}} \circ \sigma$ over $\mathrm{Spec}(W(R^+)/[\varpi])$. We can see that $\varphi^*\sigma \circ \Phi = \Phi_{\mathcal{D}} \circ \sigma$ over $\mathrm{Spec}(B^{A^+}_{[r,\infty]}/[\varpi])$ and in particular the morphism $\varphi^*\sigma : \varphi^*\mathscr{T} \to \mathcal{D}$ is ϕ-equivariant over this locus. By lemma 2.1.27 there is a unique isogeny over $\mathcal{Y}^{A^+}_{[r,\infty]}$ denoted $\lambda : \varphi^*\mathscr{T} \to \mathcal{D}$ such that $\lambda = \varphi^*\sigma$ when restricted to $\mathrm{Spec}(B^{A^+}_{[r,\infty]}/[\varpi])$. We can associate to our original data:

$$(\mathscr{T}, \psi, \sigma) \mapsto (\varphi^*\mathscr{T}, \Phi, \lambda, \varphi^*\sigma)$$

This construction is functorial when we let (A, A^+) vary by the uniqueness of λ. This gives a map $\Theta : \widehat{LGr}_{\mathcal{D}} \to \widehat{\mathrm{LSht}}_{\mathcal{D}}$. Moreover, we have $g \cdot (\varphi^*\mathscr{T}, \Phi, \tau, \varphi^*\sigma) = (\varphi^*\mathscr{T}, \Phi, \tau, g \circ \varphi^*\sigma)$ and $g \cdot (\mathscr{T}, \lambda, \sigma) = (\mathscr{T}, \lambda, \phi^* g \circ \sigma)$ so the map Θ is $\widehat{LG}_{\mathcal{D}}$-equivariant.

We construct explicitly the inverse Θ^{-1}. Given a tuple $(\mathscr{T}, \Phi, \lambda, \sigma) \in \widehat{\mathrm{LSht}}_{\mathcal{D}}(A, A^+)$ we can assign:

$$(\mathscr{T}, \Phi, \lambda, \sigma) \mapsto (\phi^*\mathscr{T}, \sigma \circ \Phi, \phi^*\sigma)$$

this construction is clearly functorial in (A, A^+), and if $\varpi_A \in A^+$ is such that $\lambda = \sigma$ over $B^{A^+}_{[r,\infty]}/[\varpi_A]$ then $\Phi_{\mathcal{D}} \circ \phi^*\sigma = \sigma \circ \Phi$ over $\mathrm{Spec}(W(A^+)/[\varpi_A])$ since λ is ϕ-equivariant. This gives a map $\Omega : \widehat{\mathrm{LSht}}_{\mathcal{D}} \to \widehat{LGr}_{\mathcal{D}}$ the composition $\Omega \circ \Theta$ is clearly the identity. One can verify directly that $\Theta \circ \Omega(\mathscr{T}, \Phi, \lambda, \sigma) = (\mathscr{T}, \Phi, \lambda', \sigma)$ for some λ' nevertheless $\lambda' = \sigma = \lambda$ over $B^{A^+}_{[r,\infty]}/[\varpi]$ as ϕ-equivariant maps for some $\varpi \in A^+$. By the uniqueness part of lemma 2.1.27 we have $\lambda = \lambda'$.

One can also verify directly by the construction of Θ that it preserves the boundedness condition so that $\Theta : \widehat{LGr^{\leq\mu}_{\mathcal{D}}} \to \widehat{\mathrm{LSht}^{\leq\mu}_{\mathcal{D}}}$ is also an isomorphism. Finally, we have

$$\pi_0(\widehat{\mathrm{Sht}^{\mathcal{D},\leq\mu}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}}) = \pi_0(\widehat{\mathrm{LSht}^{\leq\mu}_{\mathcal{D}}}) = \pi_0(\widehat{LGr^{\leq\mu}_{\mathcal{D}}}) = \pi_0(\widehat{\mathrm{Gr}^{\mathcal{D},\leq\mu}_{W(\mathfrak{f})}}_{/can_{\mathcal{D}}})$$

since the v-sheaf in groups $\widehat{LG}_{\mathcal{D}}$ is connected. □

Let us prove that moduli spaces of mixed characteristic shtukas are rich p-smelted kimberlites.

Theorem 2.3.18. *With the notation as in the beginning of this section we have that the map $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})} \to W(\mathfrak{f})^\diamondsuit$ forms a rich p-smelted kimberlite with connected p-adic tubular neighborhoods.*

Proof. Proposition 2.3.12 proves this map forms a p-smelted kimberlite. In [53] 23.3.3 it is proven that the period morphism $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})} \to \mathrm{Gr}^{\mathscr{G},\leq\mu}_{W(\mathfrak{f})}$ is étale. By proposition 1.4.34 and theorem 2.2.34 we know that $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f})[\frac{1}{p}]}$ is a cJ-diamond.

By theorem 1.1 of [18] we know that $X^{\mathscr{G}_b}_{\leq\mu}$ is locally Noetherian. By lemmas 1.4.43 and 1.4.44 to prove that the specialization map is a quotient and specializing map we only need to prove that for any non-Archimedean field extension $C/W(\mathfrak{f})[\frac{1}{p}]$ with C algebraically closed the specialization map of the base change $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_C}$ is surjective on closed points. It is then enough to prove that for any such C the p-adic tubular neighborhoods of $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_C}$ are non-empty and connected.

This follows from combining theorems 2.2.34 and 2.3.14. Indeed, if $\mathfrak{f}_C$ denotes the residue field of O_C we may apply theorem 2.3.14 to compare $(\widehat{\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f}_C)}}_{/x})_\eta$ with $(\widehat{\mathrm{Gr}^{\mathscr{G},\leq\mu}_{W(\mathfrak{f}_C)}}_{/y})_\eta$ for some y.

Since $C/W(\mathfrak{f}_C)[\frac{1}{p}]$ is purely ramified we have identifications $|(\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_C})^{\mathrm{red}}| = |(\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f}_C)})^{\mathrm{red}}|$ and $|(\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_C})^{\mathrm{red}}| = |(\mathrm{Gr}^{\mathscr{G},\leq\mu}_{W(\mathfrak{f}_C)})^{\mathrm{red}}|$. Moreover, for any $x \in |(\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_C})^{\mathrm{red}}|$ and any $y \in |(\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_C})^{\mathrm{red}}|$ we have the identities

$$(\widehat{\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_C}}_{/x})_\eta = (\widehat{\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{W(\mathfrak{f}_C)}}_{/x})_\eta \times_{W(\mathfrak{f})^\diamond} O_C^\diamondsuit,$$

and

$$(\widehat{\mathrm{Gr}^{\mathscr{G},\leq\mu}_{O_C}}_{/y})_\eta = (\widehat{\mathrm{Gr}^{\mathscr{G},\leq\mu}_{W(\mathfrak{f}_C)}}_{/y})_\eta \times_{W(\mathfrak{f})^\diamond} O_C^\diamondsuit$$

which finishes the proof of the claim. □

We finish this section with the proof of theorem 2 which is a rephrasing of 2.3.18 in less technical language. For the convenience of the reader we write the statement again.

Theorem 2.3.19. *With notation as in the introduction the following holds:*

a) There is a continuous specialization map

$$\mathrm{sp}_{\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_{F_2}}} : |\mathrm{Sht}_{(\mathscr{G},b,\mu),F_2^\diamond}| \to |X^{\mathscr{G}}_{\leq\mu}(b)|,$$

this map is a specializing and spectral map of locally spectral topological spaces. It is a quotient map and $J_b(\mathbb{Q}_p)$-equivariant.

b) Given a closed point $x \in |X^{\mathscr{G}}_{\leq\mu}(b)|$ let $S_x = \mathrm{sp}_{\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_{F_2}}}{}^{-1}(x)$, then the interior S_x° of S_x as a subspace of $|\mathrm{Sht}_{(\mathscr{G},b,\mu),F_2^\diamond}|$ is dense in S_x.

c) S_x and S_x° are non-empty and connected.

d) The specialization map induces a $J_b(\mathbb{Q}_p)$-equivariant bijection of connected components

$$\mathrm{sp}_{\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_{F_2}}} : \pi_0(\mathrm{Sht}_{(\mathscr{G},b,\mu),F_2^\diamond}) \to \pi_0(X^{\mathscr{G}}_{\leq\mu}(b))$$

Proof of theorem 1. We may apply theorem 2.3.18 and proposition 2.3.10 to conclude that the pair $(\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O^\diamond_{F_2}}, \mathrm{Sht}_{(\mathscr{G},b,\mu),F_2^\diamond})$ is a rich smelted kimberlite with reduction $X^{\mathscr{G}}_{\leq\mu}(b)$. This implies by proposition 1.4.20 that the specialization map $\mathrm{sp}_{\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_{F_2}}} : |\mathrm{Sht}_{(\mathscr{G},b,\mu),F_2^\diamond}| \to |X^{\mathscr{G}}_{\leq\mu}(b)|$ is a spectral map of locally spectral spaces. $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O^\diamond_{F_2}}$ is rich, by definition the specialization map is specializing and a quotient map. Moreover, $J_b(\mathbb{Q}_p)$ acts on $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_{F_2}}$ by ϕ-equivariant automorphisms of $\mathscr{G}_b$, since the construction of the specialization map is functorial in the category of smelted kimberlites the map is equivariant, this finishes the proof of the first clam. Let $x \in |X^{\mathscr{G}}_{\leq\mu}(b)|$, we can use proposition 1.4.29 to identify S_x° with $|(\widehat{\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_{F_2}}}_{/x})_\eta|$. Since $\mathrm{Sht}^{\mathscr{G}_b,\leq\mu}_{O_{F_2}}$ is rich we can apply proposition 1.4.33 to prove that S_x° is dense in S_x giving the second claim. By theorem 2.3.18 S_x° is connected and since it is dense in S_x this later one is also connected giving the third claim. For the last claim we may apply proposition 1.4.42. □

Chapter 3

Geometric connected components at infinite level.

3.1 Notation

Let us fix some notation for this chapter. We let k be a perfect field in characteristic p with algebraic closure $\overline{k}$. For most things the case of interest are when $k = \overline{\mathbb{F}}_p$ or when k is a finite field. In most subsections we will assume that k is algebraically closed, and we will point out when this assumption is taken. We let $W(k)$ (respectively $W(\overline{k})$) denote the ring of p-typical Witt vectors of k, respectively $\overline{k}$, and we let $K_0 = W(k)[\frac{1}{p}]$, respectively $\breve{K}_0 = W(\overline{k})[\frac{1}{p}]$. In the sections in which we assume $k = \overline{k}$ we use the symbols K_0 and $\breve{K}_0$ interchangeably.

We denote by σ the canonical lift of arithmetic Frobenious to $\breve{K}_0$ and abusing notation we will also denote by σ its restriction to K_0. We fix an algebraic closure $\overline{\breve{K}_0}$ of $\breve{K}_0$, and we let C_p denote the p-adic completion of $\overline{\breve{K}_0}$. We use K (respectively $\breve{K}$) to denote subfields of C_p of finite degree over K_0 (respectively $\breve{K}$). We let Γ_K (respectively $\Gamma_{\breve{K}}$) denote the continuous automorphisms of C_p that fix K (respectively $\breve{K}$). If $\overline{K_0}$ is the algebraic closure of K_0 in C_p then Γ_K is canonically isomorphic to $Gal(\overline{K}_0/K)$, since $\overline{K}_0$ is dense in C_p. We will denote by Γ_K^{op} the opposite group which we identify with the group of automorphisms of $\mathrm{Spec}(C_p)$ over $\mathrm{Spec}(K_0)$.

We let $W_{\breve{K}_0}$ denote the subset of continuous automorphisms of $\mathrm{Aut}(C_p)$ that stabilize $\breve{K}_0$ and act as an integral power of σ on $\breve{K}_0$. We topologize $W_{\breve{K}_0}$ so that $\Gamma_{\breve{K}_0}$ is an open subgroup. Suppose $E \subseteq C_p$ is a field of finite degree over $\mathbb{Q}_p$, and let $\mathbb{Q}_{p^s}$ be the maximal unramified extension of $\mathbb{Q}_p$ contained in E. The extension $E/\mathbb{Q}_{p^s}$ is totally ramified and $E \otimes_{\mathbb{Q}_{p^s}} \breve{K}_0$ is canonically isomorphic to the compositum $\breve{E} = E \cdot \breve{K}_0$ inside of C_p, since E and $\breve{K}_0$ are linearly disjoint and have canonical inclusions into C_p. We define an automorphism $\hat{\sigma} \in Aut(\breve{E})$ as the automorphism that maps to $Id \otimes \sigma$ under this identification. We let $W_{\breve{E}/E}$ denote the continuous automorphisms of C_p that stabilize $\breve{E}$, act on $\breve{E}$ as $\hat{\sigma}^{s\cdot n}$ for some $n \in \mathbb{Z}$. Notice that $W_{\breve{E}/E}$ fixes E.

Through out the text G will denote a connected reductive group over $\mathbb{Q}_p$. In certain subsections we will add the additional assumptions that G is quasi-split or even stronger that it is unramified over $\mathbb{Q}_p$. We will point out when one of these two assumptions are taken. Whenever G is quasi-split we will denote by A a maximally split sub-torus of G defined over $\mathbb{Q}_p$, T will denote the centralizer of A which is also a torus and B will denote a $\mathbb{Q}_p$-rational Borel containing T. If G is assumed unramified we will sometimes also assume that G is given as the basechange of a connected reductive group over $\mathbb{Z}_p$ which we will still denote by G.

We will often work in the situation in which we are given an element $b \in G(K_0)$ and/or a cocharacter $\mu : \mathbb{G}_m \to G_K$. In these circumstances $[b]$ always denotes the σ-conjugacy class of b in $G(\breve{K}_0)$ and $[\mu]$ denotes the unique geometric conjugacy class of cocharacters $[\mu] \in Hom(\mathbb{G}_m, G_{\overline{\mathbb{Q}_p}})$ that is conjugate to μ through the action of G_{C_p}. Moreover, we let $E/\mathbb{Q}_p$ denote the field extension contained in C_p over which $[\mu]$ is defined. We let E_0 denote the compositum of E and K_0 in C_p.

3.2 The geometric perspective of crystalline representations

3.2.1 Vector bundles, isocrystals and crystalline representations.

Let K_0, K and C_p be as in the notation. With this setup in [15], Fargues and Fontaine construct a remarkable $\mathbb{Q}_p$-scheme, X_{FF,C_p}, which is now known as "the fundamental curve of arithmetic".

Fargues and Fontaine justify why we can think of X_{FF,C_p} as a "curve" despite the fact that the structure morphism $\mathrm{X}_{FF,C_p} \to \mathrm{Spec}(\mathbb{Q}_p)$ is not of finite type. Moreover, the "curve" is "complete" in an appropriate sense which in particular implies that $H^0(\mathrm{X}_{FF,C_p}, \mathcal{O}_X) = \mathbb{Q}_p$. The curve comes endowed with a section "at infinity" given by a map $\infty : \mathrm{Spec}(C_p) \to \mathrm{X}_{FF,C_p}$ and it also has a $\Gamma^{op}_{K_0}$-action whose unique Γ^{op}_K-fixed point (for all finite extensions K/K_0) is ∞. The completion of the stalk of the structure sheaf at ∞, $\widehat{\mathcal{O}_{X,\infty}}$, is canonically isomorphic to Fontaine's period ring B^+_{dR} and compatibly with the Γ_{K_0}-action. Moreover, $\mathrm{X}_{FF,C_p} \setminus \infty$ is an affine scheme and $\mathrm{H}^0(\mathrm{X}_{FF,C_p} \setminus \infty, \mathcal{O}_X) = B_e = B^{\varphi=1}_{crys}$, which is a principal ideal domain. With this curve at hand Fargues and Fontaine reinterpret geometrically the classical p-adic Hodge theory of Fontaine. We recall this geometric reinterpretation for the case of crystalline representations and the connection with Scholze's theory of diamonds.

Denote by $\varphi\text{−}\mathrm{Mod}_{K_0}$ the category of isocrystals over K_0 that has as objects the pairs (D, φ) where D is a finite dimensional K_0 vector space and $\varphi : \sigma^* D \to D$ is an isomorphism. This is a $\mathbb{Q}_p$-linear Tannakian category. Fargues and Fontaine associate to $(D, \varphi) \in \varphi\text{−}\mathrm{Mod}_{K_0}$ a vector bundle $\mathcal{E}(D, \varphi)$ that comes equipped with a $\Gamma^{op}_{K_0}$-action that is compatible with the action on X_{FF,C_p} (See [15] 10.2.1, 9.1.1). By this we mean that for any $\gamma^{op} \in \Gamma^{op}_{K_0}$ inducing the associated isomorphism $\theta_{\gamma^{op}} : \mathrm{X}_{FF,C_p} \to \mathrm{X}_{FF,C_p}$ we are given a family of compatible isomorphisms

$$\Theta_{\gamma^{op}}:\theta^*_{\gamma^{op}}\mathcal{E}(D,\varphi)\to\mathcal{E}(D,\varphi).$$

The Beauville-Laszlo theorem (see [54] Lemma 5.2.9), provides us with an equivalence from the category of vector bundles over X_{FF,C_p} to the category of triples (M_e, M^+_{dR}, u) where M_e is a free module over B_e, M^+_{dR} is a free module over B^+_{dR} and $u : M_e\otimes_{B_e} B_{dR}\to M^+_{dR}\otimes_{B^+_{dR}} B_{dR}$ is an isomorphism. This is Berger's category of B-pairs. From this equivalence we get a recipe to construct vector bundles by replacing (or modifying) M^+_{dR} by some other B^+_{dR}-lattice Λ contained in $M_{dR} := M^+_{dR}\otimes_{B^+_{dR}} B_{dR}$. If we choose Λ to be stable under the action of Γ_K on M_{dR}, then the new vector bundle produced in this way will have a Γ^{op}_K-action compatible with the one on X_{FF,C_p}. Fortunately, we can understand Γ_K-stable lattices in a concrete way as we recall below.

Given a finite dimensional K vector space V we can let $\mathrm{Fil}^\bullet V$ denote a decreasing filtration of K vector spaces. If $\mathrm{Fil}^\bullet V$ satisfies $\mathrm{Fil}^i V = V$ for $i\ll 0$ and $\mathrm{Fil}^i = 0$ for $i\gg 0$, we say that $\mathrm{Fil}^\bullet V$ is a bounded filtration. To such a filtration we can associate a B^+_{dR}-lattice in $V\otimes_K B_{dR}$ denoted $\mathrm{Fil}^0(V\otimes_K B_{dR})$ and given by the formula:

$$\mathrm{Fil}^0(V\otimes_K B_{dR})=\sum_{i+j=0}\mathrm{Fil}^i V\otimes_K \mathrm{Fil}^j B_{dR}.$$

Proposition 3.2.1. *(See [15] 10.4.3) Let V be a finite dimensional vector space over K. The map that assigns to a bounded filtration $\mathrm{Fil}^\bullet V$ the B^+_{dR}-lattice $\mathrm{Fil}^0(V\otimes_K B_{dR})$ in $V\otimes_K B_{dR}$ gives a bijection between the set of bounded filtrations of V and Γ_K-stable B^+_{dR}-lattices Λ in $V\otimes_K B_{dR}$. If we let ξ denote a uniformizer of B^+_{dR} then the inverse map is given by:*

$$\mathrm{Fil}^i_\Lambda(V)=\big((\xi^i\cdot\Lambda\cap V\otimes_K B^+_{dR})/(\xi^i\cdot\Lambda\cap V\otimes_K \xi\cdot B^+_{dR})\big)^{\Gamma_K}.$$

Remark 3.2.2. *The careful reader may notice that the reference constructs $\mathrm{Fil}^i_\Lambda(V)$ in a slightly different but equivalent way. We also point out the following. Let $(a_1,\dots a_n)$ denote a decreasing sequence of integers and let $\mu:\mathbb{G}_m\to\mathrm{GL}_n$ the character defined by $\mu(t)\cdot e_i = t^{a_i}e_i$. We let $\mathrm{Fil}^\bullet_\mu(K^n)$ denote the decreasing filtration associated μ with $e_j\in\mathrm{Fil}^i_\mu$ if $a_j\geq i$. Then the B_{dR} lattice associated to Fil^i_μ is generated by $\xi^{-a_i}e_i$. Notice the change of signs! Later on we will need to keep track of this.*

Denote by $\varphi-\mathrm{ModFil}_{K/K_0}$ the category of filtered φ-modules that has as objects triples $(D,\varphi,\mathrm{Fil}^\bullet D_K)$ where (D,φ) is in $\varphi-\mathrm{Mod}_{K_0}$ and $\mathrm{Fil}^\bullet D_K$ is a bounded filtration on $D\otimes_{K_0} K$. To any triple as above Fargues and Fontaine associate a vector bundle $\mathcal{E}(D,\varphi,\mathrm{Fil}^\bullet D_K)$ equipped with a Γ^{op}_K-action compatible with the action on X_{FF,C_p}. It is constructed as a modification of $\mathcal{E}(D,\varphi)$ as follows. There is a canonical Γ_K-equivariant identification u between $D\otimes_{K_0} B_{dR}$ and the global sections of the restriction of $\mathcal{E}(D,\varphi)$ to $\mathrm{Spec}(B_{dR})$. Letting $M_e = \mathrm{H}^0(\mathrm{X}_{FF,C_p}\setminus\infty,\mathcal{E}(D,\varphi))$, $M_{dR} = D\otimes_{K_0} B_{dR}$ and $M^+_{dR} = \mathrm{Fil}^0(D_K\otimes_K B^+_{dR})$ then $\mathcal{E}(D,\varphi,\mathrm{Fil}^\bullet D_K)$ is given by (M_e, M^+_{dR}, u) under the Beauville-Laszlo equivalence.

This induces an exact and fully-faithful functor

$$\varphi-\mathrm{ModFil}_{K/K_0} \hookrightarrow \mathrm{Vec}_{\mathrm{X}_{FF,C_p}}^{\Gamma_K^{op}}$$

from the category of filtered isocrystals to the category of Γ_K^{op}-equivariant vector bundles (See [15] 10.5.3). Any object of $\mathrm{Vec}_{\mathrm{X}_{FF,C_p}}^{\Gamma_K^{op}}$ in the essential image of this functor is called a crystalline vector bundle. Moreover, when the filtered isocrystal $(D, \varphi, \mathrm{Fil}^\bullet D_K)$ is "weakly admissible" Fargues and Fontaine prove that $\mathcal{E}(D, \varphi, \mathrm{Fil}^\bullet D_K)$ is semi-stable of slope 0 (See [15] 10.5.2, 10.5.6). This in particular implies that $\mathcal{E}(D, \varphi, \mathrm{Fil}^\bullet D_K)$ without the Γ_K^{op}-action is non-canonically isomorphic to $\mathcal{O}_X^d$ for $d = dim_K(D)$ so that $\mathrm{H}^0(\mathrm{X}_{FF,C_p}, \mathcal{E}(D, \varphi, \mathrm{Fil}^\bullet D_K))$ is a d-dimensional $\mathbb{Q}_p$-vector space endowed with a continuous Γ_K-action. This construction recovers the classical functor of Fontaine $V_{cris} : \varphi-\mathrm{ModFil}_{K/K_0}^{w.a.} \to \mathrm{Rep}_{\Gamma_K}(\mathbb{Q}_p)$ that associates to a weakly admissible filtered isocrystals a crystalline representation.

Remark 3.2.3. *Since we will need this later, let us be more specific about how Γ_K acts on*

$$V := \mathrm{H}^0(\mathrm{X}_{FF,C_p}, \mathcal{E}(D, \varphi, \mathrm{Fil}^\bullet D_K)).$$

Given an element $\gamma^{op} \in \Gamma_K^{op}$ we have by definition of a Γ_K^{op}-equivariant vector bundle and by adjunction a sequence of maps

$$\mathcal{E}(D, \varphi, \mathrm{Fil}^\bullet D_K) \to \theta_{\gamma^{op},*}\theta^*_{\gamma^{op}}\mathcal{E}(D, \varphi, \mathrm{Fil}^\bullet D_K) \xrightarrow{\theta_{\gamma^{op},*}\Theta_{\gamma^{op}}} \theta_{\gamma^{op},*}\mathcal{E}(D, \varphi, \mathrm{Fil}^\bullet D_K).$$

We can pass to global sections and let $H^0(\gamma^{op}) : H^0(\mathcal{E}) = V \to V = H^0(\theta_{\gamma^{op},*}\mathcal{E})$ denote the operator obtained in this way. Notice that $\gamma^{op} \mapsto H^0(\gamma^{op})$ is contravariant and does not give a group homomorphism. But the composition of maps of sets $\Gamma_K \to \Gamma_K^{op} \to Aut(V)$ given by

$$\gamma \mapsto \mathrm{Spec}(\gamma) \mapsto H^0(\mathrm{Spec}(\gamma))$$

is a group homomorphism.

3.2.2 Families of B_{dR}-lattices

One can upgrade geometrically the situation using Scholze's theory of diamonds, since this theory allows us to consider "families" of B_{dR}^+-lattices as a geometric object. Recall that the Fargues-Fontaine curve X_{FF,C_p} has a counterpart $\mathcal{X}_{FF,C_p^\flat}$ in the category of adic spaces. Moreover it also has relative analogues. If S be an affinoid perfectoid space in characteristic p, Kedlaya and Liu (See [30] §8.7) associate to S an adic space $\mathcal{X}_{FF,S}$ that they call the relative Fargues-Fontaine curve. This construction is functorial in $\mathrm{Perf}_{\mathbb{F}_p}$, the category of affinoid perfectoid spaces in characteristic p. Moreover, if (D, φ) is an isocrystal over K_0 and S is an affinoid perfectoid space over $\mathrm{Spa}(k, k)$ one can construct a vector bundle $\mathcal{E}_S(D, \varphi)$ over $\mathcal{X}_{FF,S}$. This construction is also functorial in Perf_k and recovers $\mathcal{E}(D, \varphi)$ when $S = \mathrm{Spa}(C_p^\flat, O_{C_p^\flat})$. Strictly speaking this also requires Kedlaya-Liu's GAGA equivalence [30] 8.7.5, 8.7.7.

In the world of diamonds we have a co-equalizer diagram

$$\mathrm{Spd}(K, O_K) = Coeq(\mathrm{Spa}(C_p^\flat, O_{C_p^\flat}) \times_{\mathrm{Spd}(K,O_K)} \mathrm{Spa}(C_p^\flat, O_{C_p^\flat}) \rightrightarrows \mathrm{Spa}(C_p^\flat, O_{C_p^\flat}))$$

and we also have an identification of affinoid perfectoid spaces

$$\mathrm{Spa}(C_p^\flat, O_{C_p^\flat}) \times_{\mathrm{Spd}(K,O_K)} \mathrm{Spa}(C_p^\flat, O_{C_p^\flat}) = \underline{\Gamma_K^{op}} \times \mathrm{Spa}(C_p^\flat, C_p^{\flat,+}).$$

If we let $S_1 = \mathrm{Spa}(C_p, O_{C_p})$ and $S_2 = \underline{\Gamma_{K_0}^{op}} \times \mathrm{Spa}(C_p^\flat, O_{C_p^\flat})$ then the Galois action of $\Gamma_{K_0}^{op}$ on X_{FF,C_p} and $\mathcal{E}(D,\varphi)$ constructed by Fargues and Fontaine can be reinterpreted as glueing datum

$$\mathcal{X}_{FF,S_2} \rightrightarrows \mathcal{X}_{FF,S_1}$$

over the pair of morphisms $S_2 \rightrightarrows S_1$. Neither the Fargues-Fontaine curve as an adic spaces nor the vector bundle $\mathcal{E}(D,\varphi)$ descend to an adic space or a vector bundle over K. But as we will see one can perform some geometric constructions in this context that will make sense as geometric objects over $\mathrm{Spd}(K, O_K)$.

Now, given a perfectoid space $S \in \mathrm{Perf}_{\mathbb{F}_p}$ the data of a map $S \to \mathrm{Spd}(K_0, O_{K_0})$ induces a "section" at infinity $\infty : S^\sharp \to \mathcal{X}_{FF,S}$. This is a closed Cartier divisor as in [53] 5.3.7 and as such it has a good notion of meromorphic functions. We consider the moduli space of meromorphic modifications of $\mathcal{E}_S(D,\varphi)$ along ∞.

Definition 3.2.4. 1. *We let* $Gr(\mathcal{E}(D,\varphi))$ *denote the functor from* $\mathrm{Perf}_{\mathrm{Spd}(K_0,O_{K_0})} \to \mathrm{Sets}$ *that assigns:*

$$(S^\sharp, f) \mapsto \{((S^\sharp, f), \mathcal{V}, \alpha)\}/\cong$$

Where $(S^\sharp, f)$ *is an untilt of* S *over* $\mathrm{Spa}(K_0, O_{K_0})$*,* $\mathcal{V}$ *is a vector bundle over* $\mathcal{X}_{FF,S}$ *and* $\alpha : \mathcal{V} \dashrightarrow \mathcal{E}_S(D,\varphi)$ *is an isomorphism defined over* $\mathcal{X}_{FF,S} \setminus \infty$ *and meromorphic along* ∞*.*

2. *Let* Gr_{GL_n} *denote the functor from* $\mathrm{Perf}_{\mathbb{Q}_p} \to \mathrm{Sets}$ *that assigns:*

$$(S^\sharp, f) \mapsto \{((S^\sharp, f), \mathcal{V}, \alpha)\}/\cong$$

Where $(S^\sharp, f)$ *is an untilt of* S *over* $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$*,* $\mathcal{V}$ *is a vector bundle over* $\mathrm{Spec}(B_{dR}^+(S^\sharp))$ *and* $\alpha : \mathcal{V} \dashrightarrow \mathcal{O}^{\oplus n}$ *is an isomorphism defined over* $\mathrm{Spec}(B_{dR}(S^\sharp))$*.*

These moduli spaces are ind-proper ind-diamonds over $\mathrm{Spd}(K_0, O_{K_0})$ (and $\mathrm{Spd}(\mathbb{Q}_p, \mathbb{Z}_p)$ respectively) and after fixing a basis of D we get an identification

$$Gr_{GL_n} \times_{\mathbb{Q}_p} \mathrm{Spd}(K_0, O_{K_0}) \cong Gr(\mathcal{E}(D,\varphi))$$

(See [19] 2.12). The second space is the Beilinson-Drinfeld Grassmanian that appears in the Berkeley notes (See [54] 20.2.1).

We can re-interpret the canonical map $\mathrm{Spa}(C_p, O_{C_p}) \to \mathrm{Spa}(K_0, O_{K_0})$ that comes from thinking of K_0 as a subfield of C_p as a map $m : \mathrm{Spd}(C_p^\flat, O_{C_p^\flat}) \to \mathrm{Spd}(K_0, O_{K_0})$. The basechange

$$Gr(\mathcal{E}_S(D,\varphi)) \times_{\mathrm{Spd}(K_0,O_{K_0}),m} \mathrm{Spd}(C_p^\flat, O_{C_p^\flat})$$

gets identified through Beauville-Laszlo glueing with the moduli space that parametrizes B_{dR}^+-lattices contained in $D\otimes_{K_0} B_{dR}$. This basechange comes equipped with $\Gamma_{K_0}^{op}$-action and the set of Γ_K-invariant B_{dR}^+-lattices in $D\otimes_{K_0} B_{dR}$ are in bijection with natural transformations $\mathrm{Spd}(K,O_K)\to Gr(\mathcal{E}_S(D,\varphi))$.

Indeed, if we parametrize Γ_K-invariant lattices using filtrations as in proposition 3.2.1, then the B_{dR}^+-lattice induced by a K-filtration $\mathrm{Fil}^\bullet D_K$ allows us to construct a tuple

$$((C_p,m),\mathcal{E}(D,\varphi,\mathrm{Fil}^\bullet D_K),\alpha)$$

where α is the canonical meromorphic isomorphism

$$\alpha:\mathcal{E}(D,\varphi,\mathrm{Fil}^\bullet D_K) \dashrightarrow \mathcal{E}(D,\varphi)$$

over $\mathcal{X}_{FF,C}\setminus\infty$ coming from the construction of $\mathcal{E}(D,\varphi,\mathrm{Fil}^\bullet D_K)$ as a modification of $\mathcal{E}(D,\varphi)$. A priori this tuple only defines a map $\mathrm{Spa}(C_p^\flat,O_{C_p^\flat})\to Gr(\mathcal{E}(D,\varphi))$ but since α is Γ_K^{op}-equivariant this descends to the desired map $\mathrm{Spd}(K,O_K)\to Gr(\mathcal{E}(D,\varphi))$.

Going on with the story one defines $Gr^{adm}(\mathcal{E}(D,\varphi))\subseteq Gr(\mathcal{E}(D,\varphi))$ to be the subsheaf of tuples for which $\mathcal{V}$ is fiberwise semi-stable of slope 0. From Kedlaya-Liu's semi-continuity theorem (see [54] 22.2.1) we know that this defines an open subfunctor which is called the admissible locus. Additionally, a map $\mathrm{Spd}(K,O_K)\to Gr(\mathcal{E}(D,\varphi))$ factors through $Gr^{adm}(\mathcal{E}(D,\varphi))$ if and only if it is coming from a weakly admissible filtration. A very remarkable aspect of the situation is that if $n=dim_{K_0}(D)$ then $Gr^{adm}(\mathcal{E}(D,\varphi))$ admits a pro-étale $\underline{\mathrm{GL}_n(\mathbb{Q}_p)}$-local system $\mathbb{L}$ that "interpolates" between the n-dimensional crystalline representations associated to (D,φ) (See [19] 2.14). Also See [38] for background on quasi-pro-étale local systems.

Remark 3.2.5. *To be more specific, a pro-étale local system $\mathbb{L}'$ on $\mathrm{Spd}(K,O_K)$ corresponds to a local system $\mathbb{L}'_{C_p^\flat}$ over $\mathrm{Spa}(C_p^\flat,O_{C_p^\flat})$ together with descent data along $\underline{\Gamma_K^{op}}\times\mathrm{Spa}(C_p^\flat,O_{C_p^\flat})\rightrightarrows\mathrm{Spa}(C_p^\flat,O_{C_p^\flat})$. But pro-étale local systems over $\mathrm{Spa}(C_p^\flat,O_{C_p^\flat})$ are trivial and of the form $\mathbb{L}'_{C_p^\flat}=\underline{\mathbb{Q}_p}\otimes_{\mathbb{Q}_p}V$ for a $\mathbb{Q}_p$-vector space V. Descent datum will correspond to giving for any $\gamma^{op}\in\underline{\Gamma_K^{op}}$ an isomorphism $\Theta_{\gamma^{op}}:\gamma^{op,*}\mathbb{L}'_{C_p^\flat}\to\mathbb{L}'_{C_p^\flat}$ in a continuous way. By adjunction and passing to global sections as in remark 3.2.3 we get a Γ_K-representation with values on $\mathrm{GL}(V)$.*

The precise claim that we will use is the following.

Proposition 3.2.6. *If $\mathrm{Fil}^\bullet D_K$ is a weakly admissible filtration of (D,φ) and*

$$\iota:\mathrm{Spd}(K,O_K)\to Gr^{adm}(\mathcal{E}(D,\varphi))$$

is the map associated to $\mathrm{Fil}^\bullet D_K$*, then* $\iota^*\mathbb{L}$ *is isomorphic to* $V_{cris}(D, \varphi, \mathrm{Fil}^\bullet)$ *when we regard* $\iota^*\mathbb{L}$ *as a continuous* Γ_K*-representation.*

Proof. This follows from the definition of the local system $\mathbb{L}$ through Kedlaya-Liu's equivalence [54] 22.3.1, from the definition of the representation associated to a pro-étale local system discussed in remark 3.2.5 and from the compatibility discussed in remark 3.2.3 together with the paragraph preceding it. $\square$

Remark 3.2.7. *In a computation done below a change of sign will appear. In this remark we discuss why this change of sign appears in a simple case. Let the notation be as in proposition 3.2.6, let* $n = dim(D)$ *and let* $V = V_{cris}(D, \varphi, \mathrm{Fil}^\bullet)$*. If we fix a trivialization* $\alpha : \mathbb{Q}_p^n \to V$ *we may conjugate the action of* Γ_K *on* V *by* α *to obtain a continuous map that we denote*

$$\rho_{H^0,\alpha} : \Gamma_K \to \mathrm{GL}_n(\mathbb{Q}_p).$$

Now, let $Triv(\iota^*\mathbb{L})$ *denote the moduli space of trivializations of* $\iota^*\mathbb{L}$*. It is a* $\mathrm{GL}_n(\mathbb{Q}_p)$ *right torsor over* $\mathrm{Spd}(K, O_K)$*. The basechange* $Triv(\iota^*\mathbb{L})_{C_p}$ *receives a semi-linear action by* Γ_K^{op} *that we can express as:*

$$\gamma^{op} : Triv(\iota^*\mathbb{L}) \times_{\mathrm{Spd}(K,O_K)} \mathrm{Spd}(C_p, O_{C_p}) \xrightarrow{(id,\gamma^{op})} Triv(\iota^*\mathbb{L}) \times_{\mathrm{Spd}(K,O_K)} \mathrm{Spd}(C_p, O_{C_p}).$$

The topological space $|Triv_{C_p}(\iota^*\mathbb{L})|$ *becomes a free* $\mathrm{GL}_n(\mathbb{Q}_p)$ *right torsor. An element* $\alpha \in Triv(\iota^*\mathbb{L})(C_p)$ *defines a unique point* $|\alpha| \in |Triv(\iota^*\mathbb{L})_{C_p}|$*. By functoriality of* $|\cdot|$ *we obtain an element* $\gamma^{op}(|\alpha|) \in |Triv(\iota^*\mathbb{L})_{C_p}|$*. Since* $\mathrm{GL}_n(\mathbb{Q}_p)$ *acts simply transitively there is a unique element* $g^\alpha_{\gamma^{op}} \in \mathrm{GL}_n(\mathbb{Q}_p)$ *with* $\gamma^{op}(|\alpha|) = |\alpha| \cdot g^\alpha_{\gamma^{op}}$ *this defines a group homomorphism*

$$\rho_{|\cdot|,\alpha} : \Gamma_K^{op} \to \mathrm{GL}_n(\mathbb{Q}_p).$$

The careful readers should convince themselves that

$$\rho_{H^0,\alpha} = \rho_{|\cdot|,\alpha} \circ (-)^{Spd,-1}$$

where $(-)^{Spd,-1} : \Gamma_K \to \Gamma_K^{op}$ *is the group isomorphism* $\gamma \mapsto \mathrm{Spd}(\gamma^{-1})$*.*

3.2.3 Isocrystals with G-structure.

We keep the notation as above, we let G denote a connected reductive group over $\mathbb{Q}_p$ and $\mathrm{Rep}_G(\mathbb{Q}_p)$ denote the Tannakian category of $\mathbb{Q}_p$-linear algebraic representations of G. Recall the following definition:

Definition 3.2.8. *(See [35] §3) An isocrystal with* G*-structure* $\mathcal{F}$*, is a* $\otimes$*-exact functor* $\mathcal{F} : \mathrm{Rep}_G(\mathbb{Q}_p) \to \varphi{-}\mathrm{Mod}_{K_0}$*.*

To an element $b \in G(K_0)$ and a representation $(V, \rho) \in \mathrm{Rep}_G(\mathbb{Q}_p)$ we associate the isocrystal

$$(D_{b,\rho}, \varphi_{b,\rho}) := (V \otimes K_0, \rho(b) \cdot (Id \otimes \sigma)),$$

ranging this construction over (V, ρ) defines an isocrystal with G-structure

$$\mathcal{F}_b : \mathrm{Rep}_G(\mathbb{Q}_p) \to \varphi\text{−}\mathrm{Mod}_{K_0}.$$

We say that two elements $b_1, b_2 \in G(K_0)$ are σ-conjugate to each other if $b_1 = g^{-1} \cdot b_2 \cdot \sigma(g)$ for some element $g \in G(K_0)$. This defines an equivalence relation and b_1 is σ-conjugate to b_2 if and only if $\mathcal{F}_{b_1}$ is isomorphic to $\mathcal{F}_{b_2}$.

Now, when $k = \overline{k}$ the set of equivalence classes of σ-conjugacy is the set $B(G)$ defined and studied by Kottwitz (See [35] §1.4). In this case, every isocrystal with G-structure is isomorphic $\mathcal{F}_b$ for some $b \in G(\breve{K}_0)$ and consequently $B(G)$ parametrizes isomorphism classes of isocrystals with G-structure. The key input in this case is Steinberg's theorem which shows the vanishing of the Galois cohomology set $H^1(\Gamma_{\breve{K}_0}, G(\breve{K}_0))$ (See [57]). The set $B(G)$ has a very rich theory, we recall some of it below. For the rest of this subsection, we will carry the assumption that $k = \overline{k}$, so that $K_0 = \breve{K}_0$.

Recall that the category of isocrystals over K_0 is semisimple and the simple objects can be parametrized by rational numbers $\lambda \in \mathbb{Q}$. In particular, every object $(D, \varphi) \in \varphi\text{−}\mathrm{Mod}_{K_0}$ admits a canonical "slope" decomposition

$$(D, \varphi) = \bigoplus_{\lambda \in \mathbb{Q}} (D_\lambda, \varphi_\lambda).$$

If we let ω_b denote the composition $\mathrm{Forg} \circ \mathcal{F}_b$ where

$$\mathrm{Forg} : \varphi\text{−}\mathrm{Mod}_{K_0} \to \mathrm{Vec}(K_0)$$

denotes the forgetful functor to the category of vector spaces over K_0, then the slope decomposition defines $\otimes$-exact $\mathbb{Q}$-grading of ω_b. In turn, this grading can be interpreted as a slope morphism $\nu_b : \mathbb{D} \to G_{K_0}$ of pro-algebraic groups, where $\mathbb{D}$ is the pro-torus with character set $X^*(\mathbb{D}) = \mathbb{Q}$.

Consider the abstract group defined as a semi-direct product $G(K_0) \rtimes \sigma \cdot \mathbb{Z}$ where σ has its natural action on $G(K_0)$.

Definition 3.2.9. *(See [44] 1.8) For an element $b \in G(K_0) = G(\breve{K}_0)$ with conjugacy class $[b] \in B(G)$ we say that:*

1. *b is decent if there exists an integer s such that $(b\sigma)^s = (s \cdot \nu_b)(p)\sigma^s$ as elements of $G(K_0) \rtimes \sigma \cdot \mathbb{Z}$.*
2. *We say that b is basic if the map $\nu_b : \mathbb{D} \to G_{K_0}$ factors through the center of G.*
3. *We say that $[b] \in B(G)$ is basic if all (equivalently some) element of $[b]$ is basic.*

Since we are assuming $k = \overline{k}$ and that G is connected reductive, every σ-conjugacy class $[b] \in B(G)$ contains a decent element (See [44] 1.11).

Assume for the rest of the subsection that G is quasi-split over $\mathbb{Q}_p$, and fix subgroups $A \subseteq T \subseteq B \subseteq G$ as in the notation section.

For $b \in G(K_0)$ we can let ν_b^{dom} denote the unique map $\nu_b^{dom} : \mathbb{D} \to T_{K_0}$ in the conjugacy class of ν_b that is dominant with respect to B. The map ν_b^{dom} factors through A and is defined over $\mathbb{Q}_p$, so we can write $\nu_b^{dom} \in X_*^+(A)_{\mathbb{Q}} = (X_*^+(T_{\overline{\mathbb{Q}_p}}) \otimes_{\mathbb{Z}} \mathbb{Q})^{\Gamma_{\mathbb{Q}_p}}$ (See [55] 3.2, Introduction of [9]). This gives a well defined map $\mathcal{N} : B(G) \to X_*^+(A)_{\mathbb{Q}}$ usually referred to as the Newton map.

Recall Borovoi's algebraic fundamental group $\pi_1(G)$ which can be defined as the quotient of $X_*(T_{\overline{\mathbb{Q}_p}})$ by the co-root lattice. This group comes equipped with $\Gamma_{\mathbb{Q}_p}$ action and Kottwitz constructs a map $\kappa_G : B(G) \to (\pi_1(G))_{\Gamma_{\mathbb{Q}_p}}$ that is usually referred to as the Kottwitz map.

An important result of Kottwitz [35] states that the map of sets

$$(\nu_b^{dom}, \kappa_G) : B(G) \to \mathcal{N} \times \pi_1(G)_{\Gamma_{\mathbb{Q}_p}}$$

is injective. This says that these invariants completely determine the isomorphism classes of isocrystals with G-structure. Now, if we are given an element $\mu \in X_*(T_{\overline{\mathbb{Q}_p}})$ with reflex field E we may define an element

$$\overline{\mu} \in X_*^+(A)_{\mathbb{Q}} = X_*^+(T_{\overline{\mathbb{Q}_p}})_{\mathbb{Q}}^{\Gamma_{\mathbb{Q}_p}}$$

by averaging over the dominant elements inside a conjugacy class in the Galois orbit of μ:

$$\overline{\mu} = \frac{1}{[E : \mathbb{Q}_p]} \sum_{\gamma \in Gal(E/\mathbb{Q}_p)} \mu^{\gamma}$$

We can now recall Kottwitz' definition of the set $B(G, \mu) \subseteq B(G)$.

Definition 3.2.10. *The set $B(G,\mu)$ consists of those conjugacy classes $[b] \in B(G)$ for which $\kappa_G([b]) = [\mu]$ in $\pi_1(G)_{\Gamma_{\mathbb{Q}_p}}$ and for which $\overline{\mu} - \nu_b^{dom} \in X_*^+(A)_{\mathbb{Q}}$ is a non-negative $\mathbb{Q}$-linear combination of positive co-roots.*

3.2.4 G-bundles and G-valued crystalline representations

In this subsection we assume again that k is perfect but not necessarily algebraically closed. We also assume that G is reductive over $\mathbb{Q}_p$ but not necessarily quasi-split over $\mathbb{Q}_p$. Just as in the case of schemes, one has a theory of G-bundles over the relative Fargues-Fontaine curve that uses a Tannakian approach (See [53] Appendix to lecture 19 for the details). Given $S \in \text{Perf}_k$ and $\mathcal{F} : \text{Rep}_G(\mathbb{Q}_p) \to \varphi\text{−Mod}_{K_0}$ an isocrystal with G-structure we can define a $\otimes$-exact functor $\mathcal{E}_{\mathcal{F},S} : \text{Rep}_G(\mathbb{Q}_p) \to Vec(\mathcal{X}_{FF,S})$ by letting

$$\mathcal{E}_{\mathcal{F},S}(V,\rho) = \mathcal{E}_S(\mathcal{F}(V,\rho)),$$

this defines a G-bundle over $\mathcal{X}_{FF,S}$. When we are given $b \in G(K_0)$ we write $\mathcal{E}_{b,S}$ instead of $\mathcal{E}_{\mathcal{F}_b,S}$. This allow us to extend Tannakianly definition 3.2.4.

Definition 3.2.11. *1. We let $Gr(\mathcal{F})$ denote the functor from $\mathrm{Perf}_{\mathrm{Spd}(K_0,O_{K_0})} \to \mathrm{Sets}$ that assigns:*

$$(S^\sharp, f) \mapsto \{((S^\sharp, f), \mathcal{G}, \alpha)\}/\cong$$

Where $(S^\sharp, f)$ is an untilt of S over $\mathrm{Spa}(K_0, O_{K_0})$, $\mathcal{G}$ is a G-bundle over $\mathcal{X}_{FF,S}$ and $\alpha : \mathcal{G} \dashrightarrow \mathcal{E}_{\mathcal{F},S}$ is an isomorphism defined over $\mathcal{X}_{FF,S} \setminus \infty$ and meromorphic along ∞. When $b \in G(K_0)$ we write $Gr(\mathcal{E}_b)$ instead of $Gr(\mathcal{F}_b)$.

2. We let Gr_G denote the functor from $\mathrm{Perf}_{\mathrm{Spd}(\mathbb{Q}_p,\mathbb{Z}_p)} \to \mathrm{Sets}$ that assigns:

$$(S^\sharp, f) \mapsto \{((S^\sharp, f), \mathcal{G}, \alpha)\}/\cong$$

Where $(S^\sharp, f)$ is an untilt of S over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, $\mathcal{G}$ is a G-bundle over $\mathrm{Spec}(B^+_{dR}(S^\sharp))$ and $\alpha : \mathcal{G} \dashrightarrow G$ is a trivialization defined over $\mathrm{Spec}(B_{dR}(S^\sharp))$.

In the previous definition the meromorphicity condition asks that for every $(V, \rho) \in \mathrm{Rep}_G(\mathbb{Q}_p)$ the associated map of vector bundles $\rho_*(\alpha) : \rho_*\mathcal{G} \dashrightarrow \mathcal{E}(D_{b,\rho}, \varphi_{b,\rho})$ is meromorphic along ∞.

As with the GL_n case, the two moduli spaces become isomorphic after basechange to $\mathrm{Spd}(K_0, O_{K_0})$. Instead of fixing a basis one has to fix an isomorphism of the fiber functors:

$$(\omega_{can} \otimes K_0) \cong \omega_{\mathcal{F}}$$

Here $\omega_{\mathcal{F}} : \mathrm{Rep}_G(\mathbb{Q}_p) \to \varphi\mathrm{-Mod}_{K_0} \to K_0 - \mathrm{Vec}$ denotes $\mathrm{Forg} \circ \mathcal{F}$, and if $b \in G(K_0)$ we write ω_b instead of $\omega_{\mathcal{F}_b}$. A careful inspection of the construction of ω_b shows that (in contrast with $\omega_{\mathcal{F}}$) there is a canonical choice of isomorphism $\omega_b \cong \omega_{can}$. We won't really use this.

As with the GL_n case we can define the admissible locus as the subsheaf $Gr^{adm}(\mathcal{E}_b) \subseteq Gr(\mathcal{E}_b)$ of those tuples $((S^\sharp, f), \mathcal{G}, \alpha)$ such that $x^*\mathcal{G}$ is the trivial G-bundle for every geometric point $x : \mathrm{Spa}(C', C'^+) \to S$. This is again an open subsheaf and it admits a pro-étale $\underline{G(\mathbb{Q}_p)}$-torsor which we will also denote by $\mathbb{L}$ (See [53] 22.5.2).

To make contact with crystalline representations one needs to recall how the Tannakian formalism interacts with filtrations, we refer the reader to [48] for the details. Recall that given a fiber functor $\omega : \mathrm{Rep}_G(\mathbb{Q}_p) \to Vec(S)$ one can consider $\otimes$-exact filtrations $\mathrm{Fil}^\bullet(\omega)$ which are sequences of $\otimes$-exact functors $\mathrm{Fil}^n(\omega) : \mathrm{Rep}_G(\mathbb{Q}_p) \to Vec(S)$ indexed by $n \in \mathbb{N}$ such that $\mathrm{Fil}^n(\omega) \supseteq \mathrm{Fil}^{n+1}(\omega)$ and that are subject to various compatibility conditions (See [48] chapitre IV §2.1.1, [11] 4.2.6). To such a filtration one can associate a $\otimes$-grading $(gr(\mathrm{Fil}^\bullet(\omega)))$ which produces a morphism of algebraic groups over S, $\mu_{\mathrm{Fil}^\bullet(\omega)} : \mathbb{G}_m \to \underline{Aut}^\otimes(\omega')$ (See [48] chapitre IV §1.3 [11] 4.2.3). Here $\omega' = (gr(\mathrm{Fil}^\bullet(\omega)))$, denotes the $\otimes$-exact functor obtained from the grading after we forget the graded structure. If $x = \mathrm{Spec}(C)$ is a geometric point of S, we may find an isomorphism $\omega'_x \cong \omega_x$ and this defines a conjugacy class of cocharacters into $\underline{Aut}^\otimes(\omega_x)$. This conjugacy class is independent of the isomorphism chosen and we can denote it $[\mu_{\mathrm{Fil}^\bullet(\omega)}(x)]$.

Now, fix an isomorphism $\omega_b \cong \omega_{can}$, we get an isomorphism $\underline{Aut}^\otimes(\omega_b) \cong G_{K_0}$. Furthermore, if we are given a conjugacy class $[\mu]$ of morphisms $\mu : \mathbb{G}_{m,\overline{K_0}} \to G_{\overline{K_0}}$ with field of definition E_0/K_0 (See [11] 6.1.2) contained in C_p, then we can consider the moduli functor of filtrations of ω_b of *type* $[\mu]$. We denote this moduli space by

$$\mathscr{F}l^{\omega_b}_{E_0,[\mu]} : \mathrm{Sch}_{/E_0} \to \mathrm{Sets},$$

it is given by the formula

$$\mathscr{F}l^{\omega_b}_{E_0,[\mu]}(R) = \left\{\mathrm{Fil}^\bullet(\omega_{b,R}) \mid [\mu_{\mathrm{Fil}^\bullet(\omega)}(x)] = [\mu]\, for\, all\, x \in \mathrm{Spec}(R)\right\}$$

where $\mathrm{Fil}^\bullet(\omega_{b,R})$ ranges over the set of $\otimes$-exact filtrations of ω_b. This functor does not depend of our choice of isomorphism $\omega_b \cong \omega_{can}$.

Since G is defined over $\mathbb{Q}_p$ the conjugacy class $[\mu]$ will be defined over a finite extension E of $\mathbb{Q}_p$ contained in C_p and $\mathscr{F}l^{\omega_b}_{E_0,[\mu]}$ is isomorphic to the basechange of a similarly defined moduli functor $\mathscr{F}l^{\omega_{can}}_{E,[\mu]}$. If F/E is a finite extension and $\mu \in [\mu]$ is a representative defined over F then μ defines a parabolic subgroup $P_\mu \subseteq G_F$ and $\mathscr{F}l^{\omega_{can}}_{F,[\mu]}$ is isomorphic to the generalized flag variety G/P_μ. In particular, $\mathscr{F}l^{\omega_{can}}_{E,[\mu]}$ and $\mathscr{F}l^{\omega_b}_{E_0,[\mu]}$ are represented by geometrically connected smooth projective schemes over $\mathrm{Spec}(E)$ and $\mathrm{Spec}(E_0)$ respectively (See [11] 6.1.4). The associated adic space $(\mathscr{F}l^{\omega_b}_{E_0,[\mu]})^{ad}$ evaluates on a complete sheafy Huber pair (R, R^+) over $\mathrm{Spa}(E_0, O_{E_0})$ to the set:

$$(\mathscr{F}l^{\omega_b}_{E_0,[\mu]})^{ad}(R, R^+) = \left\{\mathrm{Fil}^\bullet(\omega_{b,R}) \mid [\mu_{\mathrm{Fil}^\bullet(\omega)}(x)] = [\mu]\, for\, all\, x \in \mathrm{Spa}(R, R^+)\right\}$$

This description relies on theorem 2.7.7 [30] of Kedlaya and Liu, and on the fact that a morphism of adic spaces $\mathrm{Spa}(R, R^+) \to (\mathscr{F}l^{\omega_b}_{E_0,[\mu]})^{ad}$ is given by a morphism of locally ringed spaces $\mathrm{Spa}(R, R^+) \to \mathscr{F}l^{\omega_b}_{E_0,[\mu]}$ by the construction of $(\mathscr{F}l^{\omega_b}_{E_0,[\mu]})^{ad}$ ([24] 3.8). In particular, if K/K_0 is a complete non-Archimedean field extension then

$$(\mathscr{F}l^{\omega_b}_{E_0,[\mu]})^{ad}(K, O_K) = \mathscr{F}l^{\omega_b}_{E_0,[\mu]}(K).$$

Just as $[\mu]$ allows us to define $\mathscr{F}l^{\omega_b}_{E_0,[\mu]}$ it also allows us to discuss boundedness conditions for Scholze's affine B_{dR}-Grassmanians. Given an algebraically closed non-Archimedean field C in characteristic p and $C^\sharp$ an untilt over E we have an identification

$$G(B_{dR}(C^\sharp))/G(B^+_{dR}(C^\sharp) = Gr_G((C, C^+))$$

([54] 19.1.2, 19.1.1). By the Cartan decomposition we have another identification

$$G(B^+_{dR}(C^\sharp)\backslash G(B_{dR}(C^\sharp)/G(B^+_{dR}(C^\sharp) = Hom(\mathbb{G}_{m,\overline{\mathbb{Q}}_p}, G_{\overline{\mathbb{Q}}_p})/G.$$

This identification sends a conjugacy class $[\mu]$ to the double coset defined by $\xi^\mu := \mu(\xi)$ where $\xi \in B^+_{dR}(C^\sharp)$ is a uniformizer. Notice that to define the map it is crucial to have a fixed embedding $E \subseteq C^\sharp$ so that the conjugacy class of $\mu_{C^\sharp}$ is well defined.

The set of conjugacy classes of cocharacters comes equipped with a partial order called the Bruhat order. Given a map $m \in Gr_G \times_{\mathbb{Q}_p} \mathrm{Spd}(E, O_E)(R, R^+)$ and a geometric point

$x : \mathrm{Spa}(C, C^+) \to \mathrm{Spa}(R, R^+)$ we say that m has relative position of type $[\mu]$ at x, (of type $\leq [\mu]$ at x respectively), if the pullback x^*m lands in the double coset associated to $[\mu]$ (a coset bounded by $[\mu]$ respectively). This allow us to define subsheaves

$$Gr^{[\mu]}_{G,E} \subseteq Gr^{\leq [\mu]}_{G,E} \subseteq Gr_G \times \mathrm{Spd}(E, O_E),$$

given by the condition that for every geometric point, the pullback x^*m has relative position $[\mu]$ (bounded by $[\mu]$ respectively). The space $Gr^{\leq [\mu]}_{G,E}$ is spatial diamond that is proper over $\mathrm{Spd}(E, O_E)$ and $Gr^{[\mu]}_{G,E} \subseteq Gr^{\leq [\mu]}_{G,E}$ is an open subdiamond.

We can now compare the affine B_{dR}-Grassmanian to the flag variety. Recall that there is a Tannakianly defined Bialynicki-Birula map ([54] 19.4.2),

$$\pi^{[\mu]}_{BB} : Gr^{[\mu]}_{G,E} \to (\mathscr{F}l^{\omega_{can}}_{E,[-\mu]})^{\diamond}.$$

We emphasize that there is a change of signs which is a consequence of the change of signs that appeared in remark 3.2.2 and of our convention on filtrations. Let us sketch the construction of this map. Let $m \in Gr^{[\mu]}_{G,E}(R, R^+)$ and let $(V, \rho) \in \mathrm{Rep}_G(\mathbb{Q}_p)$ be a representation. Then $\rho_*(m) \in Gr^{[\rho\circ\mu]}_{GL_n,E}(R, R^+)$ is a tuple $((R^\sharp, f), \mathcal{V}_{\rho,m}, \alpha_{\rho,m})$ where $\mathcal{V}_{\rho,m}$ is a projective $B^+_{dR}(R^\sharp)$-module and $\alpha_{\rho,m}$ an isomorphism of the form:

$$\alpha_{\rho,m} : \mathcal{V}_{\rho,m} \otimes_{B^+_{dR}} B_{dR}(R^\sharp) \to V \otimes_E B_{dR}(R^\sharp)$$

Let $\Lambda_{\rho,m}$ denote $\alpha_{\rho,m}(\mathcal{V}_{\rho,m}) \subseteq V \otimes_E B_{dR}(R^\sharp)$ and identify $V \otimes_E R^\sharp$ with

$$(V \otimes_E B^+_{dR}(R^\sharp))/\xi \cdot (V \otimes_E B^+_{dR}(R^\sharp)).$$

We let

$$\mathrm{Fil}^i_{\rho,m}(V \otimes_E R^\sharp) = (\xi^i \cdot \Lambda_{\rho,m} \cap V \otimes_E B^+_{dR}(R^\sharp))/(\xi^i \cdot \Lambda_{\rho,m} \cap \xi(V \otimes_E B^+_{dR}(R^\sharp))).$$

Using the techniques discussed in ([54] 19.4.2) one can justify that each $\mathrm{Fil}^i_{\rho,m}(V \otimes_E R^\sharp)$ is a $R^\sharp$-vector sub-bundle of $V \otimes_E R^\sharp$ and that the family $\mathrm{Fil}^\bullet_m(\omega_{can})[V, \rho] := \mathrm{Fil}^\bullet_{\rho,m}(V \otimes_E R^\sharp)$ is a $\otimes$-exact filtration of ω_{can} over $R^\sharp$. Then, $\pi^{[\mu]}_{BB}(m) = \mathrm{Fil}^\bullet_m(\omega_{can})$.

Let E_0 denote the compositum of E and K_0 in C_p. With an analogous construction as the one sketched above one can also construct the following variation of the Bialynicki-Birula map

$$\pi^{[\mu]}_{BB} : Gr^{[\mu]}_{E_0}(\mathcal{E}_b) \to \mathscr{F}l^{\omega_b}_{E_0,[-\mu]}.$$

This allows the following group-theoretically enhanced rephrasing of proposition 3.2.1.

Proposition 3.2.12. *Let $b \in G(K_0)$ let $[\mu] \in Hom(\mathbb{G}_m, G_{\overline{\mathbb{Q}_p}})/G$ and let K/E_0 be a finite* *field extension. Then, the Bialynicki-Birula map induces a bijection*

$$\pi^{[\mu]}_{BB} : Gr^{[\mu]}(\mathcal{E}_b)(K, O_K) \cong (\mathscr{F}l^{\omega_b}_{E_0,[-\mu]})^{\lozenge}(K, O_K),$$

of $\mathrm{Spd}(K, O_K)$*-valued points.*

Proof. One may take a faithful representation $\rho : G \to \mathrm{GL}(V)$, this induces the following commutative diagram.

$$\begin{tikzcd}
Gr^{[\mu]}_{E_0}(\mathcal{E}_b) \arrow[r, "\pi^{[\mu]}_{BB}"] \arrow[d] & (\mathscr{F}l^{\omega_b}_{E_0,[-\mu]})^{\lozenge} \arrow[d] \\
Gr^{[\rho\circ\mu]}_{E_0}(\rho_*\mathcal{E}_b) \arrow[r, "\pi^{[\rho\circ\mu]}_{BB}"] & (\mathscr{F}l^{\omega_{\rho(b)}}_{E_0,[-\rho\circ\mu]})^{\lozenge}
\end{tikzcd}$$

In this diagram, the two vertical arrows are closed immersions. From proposition 3.2.1, and by taking into account the boundedness conditions, one can deduce that the horizontal bottom arrow induces a bijection

$$\pi^{[\rho\circ\mu]}_{BB} : Gr^{[\rho\circ\mu]}_{E_0}(\rho_*\mathcal{E}_b)(K, O_K) \to (\mathscr{F}l^{\omega_{\rho(b)}}_{E_0,[-\rho\circ\mu]})^{\lozenge}(K, O_K).$$

Clearly the top horizontal arrow is injective since the vertical arrows will induce injections on (K, O_K)-points.

To prove surjectivity let $m \in (\mathscr{F}l^{\omega_b}_{E_0,[-\mu]})^{\lozenge}(K, O_K)$. We may use that the construction of proposition 3.2.1 and the Beauville-Laszlo theorem are functorial to produce from m a Γ_K^{op}-equivariant modification of G-bundles

$$\alpha : \mathcal{G} \dashrightarrow \mathcal{E}_{b,C_p}.$$

This induces an element $n : \mathrm{Spd}(K, O_K) \to Gr^{[\mu]}_{E_0}(\mathcal{E}_b)$ with $\pi^{[\mu]}_{BB}(n) = m$. □

Let $\mathrm{Rep}^{cont}_{\Gamma_K}(\mathbb{Q}_p)$ denote the category of continuous Galois representations. It is a neutral Tannakian category with canonical fiber functor $\omega^{\Gamma_K}_{can}(W, \tau) = W$. Recall that by the Tannakian formalism to specify a continuous representation $\rho : \Gamma_K \to G(\mathbb{Q}_p)$ (up to $G(\mathbb{Q}_p)$-conjugation) it is sufficient to specify a $\otimes$-exact functor $\mathcal{F} : \mathrm{Rep}_{\mathrm{G}}(\mathbb{Q}_p) \to \mathrm{Rep}^{cont}_{\Gamma_K}(\mathbb{Q}_p)$ for which $\omega^{\Gamma_K}_{can} \circ \mathcal{F}$ is isomorphic to ω_{can}. Now, the full subcategory $\mathrm{Rep}^{crys}_{\Gamma_K}(\mathbb{Q}_p)$ of crystalline representations is Tannakian and we can define crystalline representations with G-structure as those $\otimes$-exact functors $\mathcal{F} : \mathrm{Rep}_G(\mathbb{Q}_p) \to \mathrm{Rep}^{cont}_{\Gamma_K}(\mathbb{Q}_p)$ such that $\mathcal{F}(V, \rho)$ is crystalline for all $(V, \rho) \in \mathrm{Rep}_G(\mathbb{Q}_p)$.

Given a pair (b, μ) with $b \in G(K_0)$ and $\mu : \mathbb{G}_{m,K} \to G_K$ we can construct a filtered isocrystal with G-structure by defining a functor

$$\mathcal{F}_{b,\mu} : \mathrm{Rep}_G(\mathbb{Q}_p) \to \varphi\text{−}\mathrm{ModFil}_{K/K_0}$$

such that

$$\mathcal{F}_{b,\mu}(V,\rho) = (D_{b,\rho}, \varphi_{b,\rho}, \mathrm{Fil}^\bullet_\mu)$$

with

$$\mathrm{Fil}^i_\mu(D_{b,\rho}\otimes K) = \oplus_{n\leq i}(V\otimes K)^{(\rho\circ\mu(t)\cdot v = t^n\cdot v)}.$$

Definition 3.2.13. *(See [44] 1.18). We say that a pair (b,μ) with $b\in G(K_0)$ and $\mu:\mathbb{G}_m\to G_K$ is admissible if the functor $\mathcal{F}_{b,\mu}$ only takes values on weakly admissible filtered isocrystals.*

In general, even if (b,μ) is admissible the functor $V_{cris}\circ\mathcal{F}_{b,\mu}$ might not define a crystalline representation with G-structure. Indeed, the composition $\omega_{can}^{\Gamma_K}\circ V_{cris}\circ\mathcal{F}_{b,\mu}$ might fail to be isomorphic to ω_{can}. Nevertheless, this issue goes away if we impose that $[b]$, the σ-conjugacy class of b in $G(\breve{K}_0)$, lies on the Kottwitz set $B(G,\mu)$ (See [11] 11.4.3).

Associated to the admissible pair (b,μ) there is a map $y_{b,\mu}:\mathrm{Spd}(K,O_K)\to \mathscr{F}l^{\omega_b}_{E_0,[-\mu]}$ defined by the filtration $\mathrm{Fil}^\bullet_\mu$ on ω_b, and we can let $x_{b,\mu}:\mathrm{Spd}(K,O_K)\to Gr^{[\mu]}_{E_{0\mu}}(\mathcal{E}_b)$ denote the unique lift of $y_{b,\mu}$ of proposition 3.2.12. The following is a group-theoretic refinement of proposition 3.2.6 and it is one of the key inputs from modern p-adic Hodge theory that we will need later on.

Proposition 3.2.14. *Suppose that (b,μ) is an admissible pair with $[b]\in B(G,\mu)$, then the map $x_{b,\mu}:\mathrm{Spd}(K,O_K)\to Gr^{[\mu]}_{E_0}(\mathcal{E}_b)$ factors through the admissible locus $Gr^{[\mu],adm}_{E_0}(\mathcal{E}_b)$. Moreover, if $\mathbb{L}$ denotes the pro-étale $\underline{G(\mathbb{Q}_p)}$-torsor on $Gr^{adm}(\mathcal{E}_b)$ then $x^*_{b,\mu}\mathbb{L}$ agrees with the crystalline representation with G-structure defined by the functor $V_{cris}\circ\mathcal{F}_{b,\mu}$.*

Proof. Let $(V,\rho)\in\mathrm{Rep}(\mathbb{Q}_p)$ and consider the Γ_K-equivariant modification

$$\alpha:\mathcal{V}_{x_{(b,\mu)},\rho}\dashrightarrow\mathcal{E}_{b,C_p}(V,\rho)$$

associated to $\rho\circ x_{b,\mu}\in Gr(\mathcal{E}_b(V,\rho))(K,O_K)$. The admissibility of (b,μ) implies that $\mathcal{V}_{x_{(b,\mu)},\rho}$ is a semi-stable vector bundle of slope 0. Moreover, by proposition 3.2.6 there is a canonical identification

$$H^0(\mathrm{X}_{FF,C_p},\mathcal{V}_{x_{(b,\mu)},\rho}) = V_{cris}\circ\mathcal{F}_{b,\mu}(V,\rho).$$

Since $\mathcal{V}_{x_{(b,\mu)},\rho}$ is semi-stable of slope 0 we have the identification

$$\mathcal{V}_{x_{(b,\mu)},\rho} = \mathcal{O}_{\mathrm{X}_{FF,C_p}}\otimes H^0(\mathrm{X}_{FF,C_p},\mathcal{V}_{x_{(b,\mu)},\rho}).$$

Since $[b]\in B(G,\mu)$ then $\omega_{can}^{\Gamma_K}\circ V_{cris}\circ\mathcal{F}_{b,\mu}(V,\rho)\cong\omega_{can}$, and the functor

$$\mathcal{V}_{x_{(b,\mu)},-}:\mathrm{Rep}_G(\mathbb{Q}_p)\to Vec_{\mathrm{X}_{FF,C_p}}$$

is isomorphic to $\omega_{can}(-)\otimes\mathcal{O}_{\mathrm{X}_{FF,C_p}}$. Which says precisely that the G-torsor induced by a geometric point over $x_{(b,\mu)}$ is the trivial G-torsor so that $x_{(b,\mu)}$ lies in the admissible locus.

For the last part of the statement we may reason as in 3.2.5 by observing that quasi-pro-étale $\underline{G(\mathbb{Q}_p)}$-local systems over $\mathrm{Spd}(C_p^\flat, O_{C_p^\flat})$ are trivial and that $x^*_{b,\mu}\mathbb{L}$ can be interpreted as descent datum, which in turn can be interpreted as continuous Galois representations. Using the identity

$$H^0(\mathrm{X}_{FF,C_p}, \mathcal{V}_{x_{(b,\mu)},\rho}) = V_{cris} \circ \mathcal{F}_{b,\mu}(V, \rho)$$

one can justify that we get the correct Galois representation. □

3.2.5 M. Chen's result on p-adic Hodge Theory

In this subsection we assume that $k = \overline{k}$ so that $K_0 = \breve{K}_0$, we also assume that G is an unramified reductive group over $\mathbb{Q}_p$. In this case the group is quasi-split and we may choose groups $A \subseteq T \subseteq B \subseteq G$ as we have done in the notation.

Definition 3.2.15. *(See [8] 5.0.4, [9] 2.5.6) Recall the notation of definition 3.2.10. We say that a pair $([b], [\mu])$ with $[b] \in B(G, \mu)$ and $[\mu] \in X_*(T_{\overline{\mathbb{Q}_p}})$ is HN-irreducible if all the coefficients of $\overline{\mu} - \nu_b^{dom}$ as a $\mathbb{Q}$-linear combination of simple coroots are strictly positive.*

In section §4 the following result of M. Chen will be a key ingredient.

Theorem 3.2.16. *(See [8] 5.0.6)*
Let $\mu : \mathbb{G}_m \to G_K$ be a morphism and let $b \in G(K_0)$ be a decent element such that $[b] \in B(G, \mu)$ and $[\mu]$ has reflex field E. Suppose that the map $\mathrm{Spec}(K) \to \mathscr{F}l^{\omega_b}_{\breve{E},[-\mu]}$ induced by the filtration defined by μ maps to the generic point of $|\mathscr{F}l^{\omega_{can}}_{E,[-\mu]}|$ under the map

$$\mathscr{F}l^{\omega_b}_{\breve{E},[-\mu]} = \mathscr{F}l^{\omega_{can}}_{E,[-\mu]} \times_E \breve{E} \to \mathscr{F}l^{\omega_{can}}_{E,[-\mu]},$$

induced from the canonical isomorphism $\omega_{can} \otimes_{\mathbb{Q}_{p^s}} K_0 \cong \omega_b$. Assume further that the pair $([b], [\mu])$ is HN-irreducible, then the following hold:

1. *The pair (b, μ) is admissible and defines a crystalline representation $\xi_{b,\mu} : \Gamma_K \to G(\mathbb{Q}_p)$, well-defined up to conjugation.*
2. *The Zariski closure of $\xi_{b,\mu}(\Gamma_K) \subseteq G$ contains G^{der} and $\xi_{b,\mu}(\Gamma_K)$ contains an open subgroup of $G^{der}(\mathbb{Q}_p)$.*

Remark 3.2.17. *M. Chen's result is slightly stronger, but this is the formulation that we will use below. Observe that K has infinite transcendence degree over E, so it makes sense for a K-point to lie topologically over the generic point of $\mathscr{F}l^{\omega_{can}}_{E,[-\mu]}$.*

Combining proposition 3.2.14 with Chen's theorem 3.2.16 and using the fact that every element $b \in G(K_0)$ is σ-conjugate to a decent one we can deduce the following statement.

Corollary 3.2.18. *Let $b \in G(K_0)$ and $\mu \in X_*^+(T_{\overline{\mathbb{Q}_p}})$. Suppose that $[b] \in B(G, \mu)$. For every finite extension K/K_0 there is a map $x : \mathrm{Spd}(K, O_K) \to Gr(\mathcal{E}_b)^{[\mu],adm}_E$ such that if $\rho_x : \Gamma_K \to G(\mathbb{Q}_p)$ denotes the Galois representation associated to $x^*\mathbb{L}$, then $\rho_x(\Gamma_K) \cap G^{der}(\mathbb{Q}_p)$ is open in $G^{der}(\mathbb{Q}_p)$.*

3.2.6 The geometric realization of $\mathbb{L}$ and p-adic shtukas

In this section we assume $k = \overline{k}$ and we let G be any reductive group over $\mathbb{Q}_p$. We fix $b \in G(K_0)$, $[\mu] \in Hom(\mathbb{G}_m, G_{\overline{\mathbb{Q}_p}})$ and we let $E_0 = K_0 \cdot E$ denote the field of definition of $[\mu]$ over K_0. Let $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ denote an open compact subgroup, recall the moduli space of p-adic shtukas that appears in the Berkeley notes.

Definition 3.2.19. *(See [53] 23.3.1) We define* $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} : \mathrm{Perf}_k \to \mathrm{Sets}$ *as the presheaf that assigns to* $S \in \mathrm{Perf}_k$ *isomorphism classes of tuples*

$$((S^\sharp, f), \mathcal{E}, \alpha, \mathbb{P}_\mathcal{K}, \iota)$$

such that:

1. $(S^\sharp, f)$ *is an untilt of* S *over* E_0.
2. $\mathcal{E}$ *is a* G*-bundle on the relative Fargues-Fontaine* $\mathcal{X}_{FF,S}$ *curve whose fibers on geometric points of* S *are isomorphic to the trivial* G*-torsor.*
3. $\alpha : \mathcal{E} \dashrightarrow \mathcal{E}_b$ *is a modification of* G*-bundles defined over* $\mathcal{X}_{FF,S} \setminus S^\sharp$ *meromorphic along* $S^\sharp$ *and whose type is bounded by* $[\mu]$ *on geometric points.*
4. $\mathbb{P}_\mathcal{K}$ *is a pro-étale* $\underline{\mathcal{K}}$*-torsor and* ι *is an identification of* $\mathbb{P}_\mathcal{K} \times^{\mathcal{K}} \underline{G(\mathbb{Q}_p)}$ *with the pro-étale* $\underline{G(\mathbb{Q}_p)}$*-torsor that* $\mathcal{E}$ *defines under the equivalence of [53] theorem 22.5.2.*

It is proven in [53] that the presheaves $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}}$ are locally spatial diamonds over $\mathrm{Spd}(E_0, O_{E_0})$, and that whenever μ is a minuscule conjugacy class of cocharacters then $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}}$ is represented by the diamond associated to a smooth rigid-analytic space over $\mathrm{Spa}(E_0, O_{E_0})$. As Scholze and Weinstein prove ([53] 24.3.5) these moduli spaces are group-theoretic generalization of (the generic fiber of) Rapoport-Zink spaces. Since all of our arguments work for the general case of moduli spaces of p-adic shtukas we will not make distinction with the minuscule case.

Scholze and Weinstein construct a family of "Grothendieck-Messing" period morphisms

$$\pi_{GM,\mathcal{K}} : \mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \to Gr^{adm,\leq[\mu]}_{E_0}(\mathcal{E}_b)$$

given by the formula:

$$((S^\sharp, f), \mathcal{E}, \alpha, \mathbb{P}_\mathcal{K}, \iota) \mapsto ((S^\sharp, f), \mathcal{E}, \alpha)$$

For every $\mathcal{K}$ this gives a surjective étale morphism of locally spatial diamonds. Moreover, this family is functorial on $\mathcal{K}$. That is, if $\mathcal{K}_1 \subseteq \mathcal{K}_2$ are two compact and open subsets then we get a commutative diagram of étale maps,

$$\begin{tikzcd} \mathrm{Sht}_{G,b,[\mu],\mathcal{K}_1} \arrow[rr, "\pi_{\mathcal{K}_1,\mathcal{K}_2}"] \arrow[dr, "\pi_{GM,\mathcal{K}_1}"] & & \mathrm{Sht}_{G,b,[\mu],\mathcal{K}_2} \arrow[dl, "\pi_{GM,\mathcal{K}_2}"] \\ & Gr^{adm,\leq[\mu]}_{E_0}(\mathcal{E}_b) & \end{tikzcd}$$

where the transition map $\pi_{\mathcal{K}_1,\mathcal{K}_2}$ is the one deduced from assigning to $\mathbb{P}_{\mathcal{K}_1}$ the corresponding $\underline{\mathcal{K}_2}$-torsor $\mathbb{P}_{\mathcal{K}_1} \times^{\mathcal{K}_1} \mathcal{K}_2$. Also, if $\mathcal{K}_1 \subseteq \mathcal{K}_2$ is normal of finite index then the transition maps $\pi_{\mathcal{K}_1,\mathcal{K}_2}$ are surjective and finite étale.

The flexibility of the category of diamonds allows us to define moduli spaces of p-adic shtukas associated to an arbitrary compact subgroup $\mathcal{K}' \subseteq G(\mathbb{Q}_p)$ including the case $\mathcal{K}' = \{e\}$ (which is usually referred to as the infinite level). Indeed, the set of compact open subgroups $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ containing $\mathcal{K}'$ is co-filtered and has intersection equal to $\mathcal{K}'$. We may define the limit of diamonds $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}'} = \varprojlim_{\mathcal{K}' \subseteq \mathcal{K}} \mathrm{Sht}_{G,b,[\mu],\mathcal{K}}$, together with a period map

$$\pi_{GM,\mathcal{K}'} : \mathrm{Sht}_{G,b,[\mu],\mathcal{K}'} \to Gr_{E_0}^{adm,\leq[\mu]}(\mathcal{E}_b).$$

This sheaf has the structure of a locally spatial diamond. Moreover, although the period map in general might not be étale it is always a quasi-proétale map (See [51] 10.1).

Moduli spaces of shtukas at infinite level ($K' = \{e\}$) have the following pleasant description,

$$\mathrm{Sht}_{G,b,[\mu],\infty}(S) = \{(S^\sharp, f), \alpha : G \dashrightarrow \mathcal{E}_b\}$$

where $(S^\sharp, f)$ denotes an untilt of S over E_0, G denotes the trivial G-bundle over $\mathcal{X}_{FF,S}$ and α is a modification of G-bundles over $\mathcal{X}_{FF,S} \setminus S^\sharp$, meromorphic along $S^\sharp$ and whose type is bounded by $[\mu]$ on geometric points. The natural action of $G(\mathbb{Q}_p)$ on the trivial torsor G induces a right action of $\underline{G(\mathbb{Q}_p)}$ on $\mathrm{Sht}_{G,b,[\mu],\infty}$ (See §2.8 to contrast the $\underline{G(\mathbb{Q}_p)}$-action to more obvious $G(\mathbb{Q}_p)$-action). Scholze and Weinstein prove that the period map $\pi_{GM,\infty}$ together with the action of $\underline{G(\mathbb{Q}_p)}$ is the geometric realization of the pro-étale $\underline{G(\mathbb{Q}_p)}$-torsor $\mathbb{L}$ over $Gr_{E_0}^{adm,\leq[\mu]}(\mathcal{E}_b)$. In other words, they prove that the two definitions, the one given directly and the one given in terms of a limit, agree.

3.2.7 Weil descent

In this section we discuss Weil descent datum and its induced Weil-group action, for this subsection we assume $k = \overline{k}$ so that $K_0 = \breve{K}_0$. Recall that we defined $W_{\breve{E}/E}$ as the subset of continuous automorphisms of C_p that act as $\hat{\sigma} := Id_E \otimes \sigma^{n\cdot s}$ on $\breve{E} = E \cdot K_0$. It evidently contains $\Gamma_{\breve{E}}$ and we may topologize $W_{\breve{E}/E}$ so that $\Gamma_{\breve{E}} \hookrightarrow W_{\breve{E}/E}$ is a topological immersion and an open map. We get a strict exact sequence of topological groups

$$e \to \Gamma_{\breve{E}} \to W_{\breve{E}/E} \to \hat{\sigma}^{\mathbb{Z}} \to e.$$

Whenever $g \in W_{\breve{E}/E}$ we will write $g^{op} \in W_{\breve{E}/E}^{op}$ for the morphism of spaces $g^{op} : \mathrm{Spd}(C_p, O_{C_p}) \to \mathrm{Spd}(C_p, O_{C_p})$ induced by the map of fields. Note that if $g_1 = g_2 \circ g_3$ in $W_{\breve{E}/E}$ then $g_1^{op} = g_3^{op} \circ g_2^{op}$ in $W_{\breve{E}/E}^{op}$.

Definition 3.2.20. *1. Let $\mathcal{G}$ be a v-sheaf over* $\mathrm{Spd}(\breve{E}, O_{\breve{E}})$*, a Weil descent datum for $\mathcal{G}$ is an isomorphism $\tau : \mathcal{G} \to \hat{\sigma}^{op,*}\mathcal{G}$ over* $\mathrm{Spd}(\breve{E}, O_{\breve{E}})$.

2. Given Weil descent datum for $\mathcal{G}$ and $n \in \mathbb{N}$ we define inductively

$$\tau^n = \hat{\sigma}^{op,*}(\tau^{n-1}) \circ \tau : \mathcal{G} \to \hat{\sigma}^{op,*}\mathcal{G} \to \hat{\sigma}^{op,*,n}\mathcal{G}.$$

For $-n$ we define $\tau^{-n} = \hat{\sigma}^{op,*,n}([\tau^n]^{-1}) : \mathcal{G} \to \hat{\sigma}^{op,*,-n}\mathcal{G}$. We also define $\tau^0 = Id_{\mathcal{G}}$.

Weil descent datum will provide us with actions by $W^{op}_{\breve{E}}$ instead of only $\Gamma^{op}_{\breve{E}}$. In the following sections we will need to endow our spaces with continuous actions rather than plain actions by an abstract group. An efficient way to provide a v-sheaf with a continuous action is to endow it with the action of the group sheaf $\underline{W^{op}_{\breve{E}}}$ that parametrizes continuous maps $|\mathrm{Spa}(R, R^+)| \to W^{op}_{\breve{E}}$.

Lemma 3.2.21. *Suppose we are given a right $\underline{\Gamma_{\breve{E}}}$-action on a v-sheaf,*

$$m : \mathcal{F} \times \underline{\Gamma_{\breve{E}}} \to \mathcal{F},$$

and suppose we are given a group homomorphism $\theta : W^{op}_{\breve{E}} \to \mathrm{Aut}(\mathcal{F})$ such that $\theta(\gamma^{op}) = m(-, \gamma)$ for all constant elements $\gamma \in \Gamma_{\breve{E}} \subseteq \underline{\Gamma_{\breve{E}}}$. Then there is a unique right $\underline{W_{\breve{E}/E}}$-action $m' : \mathcal{F} \times \underline{W_{\breve{E}/E}} \to \mathcal{F}$ with $m'_{|\underline{\Gamma_{\breve{E}}}} = m$ and $\theta(\gamma^{op}) = m'(-, \gamma)$ for all constant elements $\gamma \in W_{\breve{E}/E}$.

Proof. Let $W^{disc}_{\breve{E}/E}$ (respectively $\Gamma^{disc}_{\breve{E}}$) denote the sheaf of locally constant maps $|\mathrm{Spa}(R, R^+)| \to W_{\breve{E}/E}$ (respectively $\Gamma_{\breve{E}}$). We observe that any element $g \in \underline{W_{\breve{E}/E}}$ can be written as $g^{disc} \cdot \gamma$ with $\gamma \in \underline{\Gamma_{\breve{E}}}$ and $g^{disc} \in W^{disc}_{\breve{E}}$. Moreover if $g^{disc}_1 \gamma_1 = g^{disc}_2 \gamma_2$ then $\gamma_1 \cdot \gamma_2^{-1} \in \Gamma^{disc}_{\breve{E}}$. To define an action of $\underline{W_{\breve{E}/E}}$ it is enough to define actions of $\underline{\Gamma_{\breve{E}}}$ and $W^{disc}_{\breve{E}}$ that agree on $\Gamma^{disc}_{\breve{E}}$ because $\underline{W_{\breve{E}/E}}(R, R^+) = W^{disc}_{\breve{E}/E}(R, R^+) \cdot \underline{\Gamma_{\breve{E}}}(R, R^+)$ and $W^{disc}_{\breve{E}}(R, R^+) \cap \underline{\Gamma_{\breve{E}}}(R, R^+) = \Gamma^{disc}_{\breve{E}}(R, R^+)$. Now, θ defines an action $m_\theta : \mathcal{F} \times W^{disc}_{\breve{E}} \to \mathcal{F}$ and the hypothesis ensure that m_θ agrees with m on $\Gamma^{disc}_{\breve{E}}$.

□

Proposition 3.2.22. *If $(\mathcal{G}, \tau)$ is a v-sheaf over $\mathrm{Spd}(\breve{E}, O_{\breve{E}})$ equipped with a Weil-descent datum then $\mathcal{G} \times_{\breve{E}} \mathrm{Spd}(C_p, O_{C_p})$ comes equipped with a right action by $\underline{W_{\breve{E}/E}}$.*

Proof. We let $\iota : \mathrm{Spd}(C_p, O_{C_p}) \to \mathrm{Spd}(\breve{E}, O_{\breve{E}})$ denote the map induced from the canonical inclusion. By lemma 3.2.21 it is enough to specify a right action by $\underline{\Gamma_{\breve{E}}}$ and a homomorphism of abstract groups $f : W^{op}_{\breve{E}} \to \mathrm{Aut}(\mathcal{G}_{C_p})$. Since $\mathcal{G}$ is defined over $\breve{E}$ and $\breve{E} = C_p/\underline{\Gamma_{\breve{E}}}$ we already have a well-defined right $\underline{\Gamma_{\breve{E}}}$-action on $\mathcal{G}_{C_p}$. Let $g \in W_{\breve{E}/E}$ restricting to $\hat{\sigma}^n$ on $\breve{E}$, we define $f(g^{op})$ as the g^{op}-linear map that appears in the top triangle of the following commutative diagram with Cartesian squares.

$$
\begin{tikzcd}
\mathcal{G}_{C_p} \arrow[r, "\iota^*\tau^n"] \arrow[drr, bend right, "f(g^{op})"'] \arrow[dd] & \iota^*(\hat{\sigma}^{op,*,n}\mathcal{G}) = g^{op,*}(\mathcal{G}_{C_p}) \arrow[rr] \arrow[dr, "g^{op}"] \arrow[dd] & & \mathrm{Spd}(C_p, O_{C_p}) \arrow[dr, "g^{op}"] \arrow[dd, "\iota"] & \\
& & \mathcal{G}_{C_p} \arrow[rr] \arrow[dd] & & \mathrm{Spd}(C_p, O_{C_p}) \arrow[dd, "\iota"] \\
\mathcal{G} \arrow[r, "\tau^n"] & \hat{\sigma}^{op,*,n}\mathcal{G} \arrow[rr] \arrow[dr] & & \mathrm{Spd}(\breve{E}, O_{\breve{E}}) \arrow[dr, "\hat{\sigma}^{op,n}"] & \\
& & \mathcal{G} \arrow[rr] & & \mathrm{Spd}(\breve{E}, O_{\breve{E}})
\end{tikzcd}
$$

Checking that f is a group homomorphism is a tedious diagram chase. To prove that the right actions of $\underline{\Gamma_{\breve{E}}}$ and $W^{disc}_{\breve{E}}$ restricted to $\Gamma^{disc}_{\breve{E}}$ are compatible we recall that the action $\underline{\Gamma_{\breve{E}}}$ on $\mathcal{G}_{C_p}$ is constructed as the limit of actions $\Gamma_{F/\breve{E}}$ on $\mathcal{G}_F$ over subfields $F \subseteq C_p$ that are Galois and of finite degree over $\breve{E}$. Each of these actions by a finite discrete group are constructed through a commutative diagram as the one above, except that for $g \in \Gamma_{F/\breve{E}}$ we have a canonical identification $\mathcal{G}_F \to g^{op,*}(\mathcal{G}_F)$. The compatibility boils down to the fact that we defined $\tau^0 = Id_{\mathcal{G}}$. $\square$

Of course given two diamonds with Weil descent datum $(\mathcal{G}_i, \tau_i)$ over $\mathrm{Spd}(\breve{E}, O_{\breve{E}})$ and a map $f : \mathcal{G}_1 \to \mathcal{G}_2$ satisfying a commutative diagram:

$$
\begin{tikzcd}
\mathcal{G}_1 \arrow[r, "f"] \arrow[d, "\tau_1"] & \mathcal{G}_2 \arrow[d, "\tau_2"] \\
\hat{\sigma}^{op,*}\mathcal{G}_1 \arrow[r, "\hat{\sigma}^{op,*}f"] & \hat{\sigma}^{op,*}\mathcal{G}_2
\end{tikzcd}
$$

the corresponding map $f : \mathcal{G}_1 \times_{\breve{E}} \mathrm{Spd}(C_p, O_{C_p}) \to \mathcal{G}_2 \times_{\breve{E}} \mathrm{Spd}(C_p, O_{C_p})$ will be $\underline{W_{\breve{E}/E}}$-equivariant.

We can give Weil descent datum to the moduli problems we have been working with.

Proposition 3.2.23.
- *There are canonical identifications of v-sheaves compatible with inclusion and with the structure map to* $\mathrm{Spd}(K_0, O_{K_0})$.
  1. $\sigma^{op,*}Gr_{K_0}(\mathcal{E}_b) = Gr_{K_0}(\mathcal{E}_{\sigma(b)})$.
  2. $\sigma^{op,*}Gr^{adm}_{K_0}(\mathcal{E}_b) = Gr^{adm}_{K_0}(\mathcal{E}_{\sigma(b)})$.
- *There are canonical isomorphisms of v-sheaves compatible with the inclusion, with the period morphism and with the structure map to* $\mathrm{Spd}(\breve{E}, O_{\breve{E}})$.
  1. $\hat{\sigma}^{op,*}Gr^{\leq[\mu]}_{\breve{E}}(\mathcal{E}_b) = Gr^{\leq[\mu]}_{\breve{E}}(\mathcal{E}_{\sigma^s(b)})$.

2. $\hat{\sigma}^{op,*}Gr_{\breve{E}}^{adm,\leq[\mu]}(\mathcal{E}_b) = Gr_{\breve{E}}^{adm,\leq[\mu]}(\mathcal{E}_{\sigma^s(b)})$.
3. $\hat{\sigma}^{op,*}\mathrm{Sht}_{G,b,[\mu],\infty} = \mathrm{Sht}_{G,\sigma^s(b),[\mu],\infty}$

Proof. Recall that $\mathrm{Spd}(K_0, O_{K_0}) = \mathrm{Spd}(k,k) \times_{\mathbb{F}_p^\diamond} \mathbb{Z}_p^\diamond$ and that $\sigma^{op} : \mathrm{Spd}(K_0, O_{K_0}) \to \mathrm{Spd}(K_0, O_{K_0})$ gets identified with $\mathrm{Frob}^{op} \times \mathrm{id}$. Given an object

$$[S \to \mathrm{Spd}(K_0, O_{K_0})] \in \mathrm{Perf}_{K_0^\diamond}$$

defined by an untilt $(S^\sharp, f)$ over $\mathrm{Spa}(K_0, O_{K_0})$ we let $S^\sigma \in \mathrm{Perf}_{K_0^\diamond}$ be given by $(S^\sharp, \sigma^{op} \circ f)$. For any sheaf $\mathcal{G}$ over $\mathrm{Spd}(K_0, O_{K_0})$ the functor $\sigma^{op,*}\mathcal{G} : \mathrm{Perf}_{K_0^\diamond} \to \mathrm{Sets}$ is given by the formula $\sigma^{op,*}\mathcal{G}(S) = \mathcal{G}(S^\sigma)$. We remark that although the construction of the relative Fargues-Fontaine curve $\mathcal{X}_{FF,S}$ does not depend on the structure map $S \to \mathrm{Spd}(k,k)$, the construction of the G-bundle $\mathcal{E}_{b,S}$ does. Actually, if $(D, \varphi) \in \varphi\text{−}\mathrm{Mod}_{K_0}$ then $\mathcal{E}_{S^\sigma}(D, \phi) = \mathcal{E}_S(\sigma^* D, \sigma^* \phi)$, and for isocrystals of the form $(D_{b,\rho}, \varphi_{b,\rho})$, with $b \in G(K_0)$ and $(V, \rho) \in \mathrm{Rep}_G(\mathbb{Q}_p)$, one can compute explicitly that

$$(\sigma^* D_{b,\rho}, \sigma^* \varphi_{b,\rho}) = (D_{\sigma(b),\rho}, \varphi_{\sigma(b),\rho}),$$

so that the equalities $\mathcal{E}_{b,S^\sigma} = \mathcal{E}_{\sigma(b),S}$ and $\mathcal{E}_{b,S^{\hat{\sigma}}} = \mathcal{E}_{\sigma^s(b),S}$ hold.

From here the proof of each item is very similar and follows from applying the formula $\sigma^{op,*}\mathcal{G}(S) = \mathcal{G}(S^\sigma)$ (or the analogous formula $\hat{\sigma}^{op,*}\mathcal{G}(S) = \mathcal{G}(S^{\hat{\sigma}})$) to the different moduli spaces. We only spell the details for $\hat{\sigma}^{op,*}Gr_{\breve{E}}^{adm,\leq[\mu]}(\mathcal{E}_b) = Gr_{\breve{E}}^{adm,\leq[\mu]}(\mathcal{E}_{\sigma^s(b)})$.

Fix $S = \mathrm{Spa}(R, R^+)$ together with a map $S \to \mathrm{Spd}(\breve{E}, O_{\breve{E}})$ and a geometric point $x : \mathrm{Spd}(C, C^+) \to S$. Recall that $\hat{\sigma} = Id \otimes \sigma^s$ so that if $\iota : E \to E \otimes_{\mathbb{Q}_{p^s}} K_0 = \breve{E}$ is the natural inclusion then $\hat{\sigma} \circ \iota = \iota$. Recall that $Gr_{\breve{E}}^{adm,\leq[\mu]}(\mathcal{E}_b)(S^{\hat{\sigma}})$ parametrizes modifications $\alpha : \mathcal{G} \dashrightarrow \mathcal{E}_{\sigma(b),S}$ with $\mathcal{G}$ fiberwise the trivial bundle and α bounded on geometric points by $[\mu]$. Now, in the preceding description we use the map $x^{\hat{\sigma}} : \mathrm{Spa}(C, C^+) \to \mathrm{Spd}(\breve{E}, O_{\breve{E}})$ to define the bijection

$$X_*(G_{\overline{\mathbb{Q}_p}})/G \cong_{x^{\hat{\sigma}}} G(B_{dR}^+(C^\sharp))\backslash G(B_{dR}(C^\sharp))/G(B_{dR}^+(C^\sharp))$$

with which we compare against μ. Notice again that the set

$$G(B_{dR}^+(C^\sharp))\backslash G(B_{dR}(C^\sharp))/G(B_{dR}^+(C^\sharp))$$

does not depend of the structure morphism $S \to \mathrm{Spd}(\breve{E}, O_{\breve{E}})$, and that the bijection only depends on the composition $x : \mathrm{Spd}(C, C^+) \to \mathrm{Spd}(E, O_E)$. Since $\hat{\sigma} \circ \iota = \iota$ we may conclude. $\square$

Now, observe that b and $\sigma(b)$ are σ-conjugate by b. More precisely, the family of linear maps

$$\rho(b) : (D_{\sigma(b),\rho}, \varphi_{\sigma(b),\rho}) \to (D_{b,\rho}, \varphi_{b,\rho})$$

is a functorial isomorphism of isocrystals that defines an isomorphism of $\otimes$-exact functors $\phi_b : \mathcal{F}_{\sigma(b)} \to \mathcal{F}_b$. The morphism of isocrystals ϕ_b extends by functoriality to morphisms of

G-bundles $\phi_b : \mathcal{E}_{\sigma(b)} \to \mathcal{E}_b$ and allows us to endow our moduli of interest with Weil descent datum, for example:

$$\tau_b : Gr_{K_0}(\mathcal{E}_b) \to \sigma^{op,*} Gr_{K_0}(\mathcal{E}_b) = Gr_{K_0}(\mathcal{E}_{\sigma(b)})$$

and

$$\tau_b : \mathrm{Sht}_{G,b,[\mu],\infty} \to \hat{\sigma}^{op,*}\mathrm{Sht}_{G,b,[\mu],\infty} = \mathrm{Sht}_{G,\sigma^s(b),[\mu],\infty}$$

by the applications

$$[((S^\sharp, f), \mathcal{G}, \alpha) \mapsto ((S^\sharp, f), \mathcal{G}, (\phi_b^{-1}) \circ \alpha)]\ [((S^\sharp, f), \mathcal{G}, \alpha) \mapsto ((S^\sharp, f), \mathcal{G}, (\phi_b^{-1})^s \circ \alpha)].$$

Moreover, it is not hard to see that the descent datum is compatible with the period morphism π_{GM}. An important feature of the situation is that the Weil descent datum on our moduli spaces only depends on the isomorphism class of the isocrystal $\mathcal{F}_b$. More precisely, if b_1 and b_2 are σ-conjugate by g, $b_1 = g^{-1}b_2\sigma(g)$ then g induces a commutative diagram like the one below

$$\begin{tikzcd}
Gr_{K_0}(\mathcal{E}_{b_1}) \arrow[r, "g"] \arrow[d, "\tau_{b_1}"] & Gr_{K_0}(\mathcal{E}_{b_2}) \arrow[d, "\tau_{b_2}"] \\
\sigma^{op,*}Gr_{K_0}(\mathcal{E}_{b_1}) \arrow[r, "\sigma^{op,*}(g)"] & \sigma^{op,*}Gr_{K_0}(\mathcal{E}_{b_2}).
\end{tikzcd}$$

Indeed, this follows from the identity $\sigma(g)b_1^{-1} = b_2^{-1}g$. The same applies to all the spaces considered in proposition 3.2.23. Using proposition 3.2.22 we can endow $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p})$ with a right $\underline{W_{\breve{E}/E}}$-action. Moreover, the space $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p})$ with its right $\underline{W_{\breve{E}/E}}$-action are independent of the choice of $b \in [b]$.

3.2.8 The action of $J_b(\mathbb{Q}_p)$

In this section we let $k = \overline{k}$. In ([35] A.2) Kottwitz shows how to associate to the $\otimes$-functor $\mathcal{F}_b : \mathrm{Rep}_G(\mathbb{Q}_p) \to \varphi-\mathrm{Mod}_{K_0}$ a connected reductive group J_b over $\mathbb{Q}_p$ whose group of $\mathbb{Q}_p$-valued points is the σ-centralizer of b,

$$J_b(\mathbb{Q}_p) = \left\{ g \in G(K_0) \mid g^{-1} \cdot b \cdot \sigma(g) = b \right\}.$$

Let us recall this construction. For any $\mathbb{Q}_p$-algebra R we let $\varphi-\mathrm{Mod}_{K_0} \otimes_{\mathbb{Q}_p} R$ denote the category whose objects are the same as in $\varphi-\mathrm{Mod}_{K_0}$ and morphisms are

$$Hom_R((D_1, \varphi_1), (D_2, \varphi_2)) := Hom_{\varphi-\mathrm{Mod}_{K_0}}((D_1, \varphi_1), (D_2, \varphi_2)) \otimes_{\mathbb{Q}_p} R$$

There is a natural $\otimes$-functor $\beta_R : \varphi-\mathrm{Mod}_{K_0} \to \varphi-\mathrm{Mod}_{K_0} \otimes_{\mathbb{Q}_p} R$ and $J_b(R)$ is defined as $Aut^{\otimes}(\beta_R \circ \mathcal{F}_b)$. With J_b defined in this way we have

$$J_b(\mathbb{Q}_p) = Aut^{\otimes}(\mathcal{F}_b) \subseteq Aut^{\otimes}(\mathrm{Forg} \circ \mathcal{F}_b) = G(K_0).$$

Moreover, recall that the slope decomposition produces a map $\nu_b : \mathbb{D} \to G_{K_0}$, if we denote M_b the centralizer of ν_b in G_{K_0} then $(J_b)_{K_0}$ is isomorphic to M_b. Since the elements of $J_b(\mathbb{Q}_p)$ act on $\mathcal{F}_b$ then we get a homomorphism of abstract groups $J_b(\mathbb{Q}_p) \to Aut(\mathcal{E}_{b,S})$ this already gives an action of $J_b(\mathbb{Q}_p)$ on $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p})$ and the other spaces we have considered, but from this description it is not clear, for example, if this action is continuous with respect to the p-adic topology on $J_b(\mathbb{Q}_p)$. A slightly better approach is to endow our moduli spaces with an action of $\underline{J_b(\mathbb{Q}_p)}$. Let us sketch how to do this following the ideas that the author learned from reading ([16] III.4.7). We point out that the reference does this in a much cleaner but less concrete way.

We let $\mathcal{J}_b : \mathrm{Perf}_{K_0^\diamond} \to \mathrm{Sets}$ denote the group sheaf that assigns to $S \to \mathrm{Spd}(K_0, O_{K_0})$ the group of automorphisms of $\mathcal{E}_{b,S}$. This is a sheaf of groups and a locally spatial diamond over $\mathrm{Spd}(K_0, O_{K_0})$. We can endow all of the moduli problems that appear in proposition 3.2.23 with an evident left action by $\mathcal{J}_b$. Moreover, it is easy to see that this action commutes with the right action of $\underline{G(\mathbb{Q}_p)}$ on $\mathrm{Sht}_{G,b,[\mu],\infty}$.

Recall that the category of isocrystals $\varphi-\mathrm{Mod}_{K_0}$ is naturally $\mathbb{Q}$-graded. This gives a family of compatible $\mathbb{Q}$-gradings on $\mathcal{E}_{b,S}(V, \rho)$ for all $(V, \rho) \in \mathrm{Rep}_G(\mathbb{Q}_p)$ and all $S \in \mathrm{Perf}_{K_0^\diamond}$. We let $J_b' \subseteq \mathcal{J}_b$ denote the subsheaf of automorphisms of $\mathcal{E}_b$ that respect the $\mathbb{Q}$-grading. In what follows we construct an injective map $\iota_b : (\underline{J_b(\mathbb{Q}_p)})_{K_0} \to \mathcal{J}_b$ of group diamonds over $\mathrm{Spd}(K_0, O_{K_0})$ that induces an isomorphism onto J_b'. We begin by explaining the vector bundle case.

Suppose (D, φ) is an isocrystal in $\varphi-\mathrm{Mod}_{K_0}$, and that

$$(D, \varphi) = \bigoplus_{\lambda \in \mathbb{Q}} (D_\lambda, \varphi_\lambda)$$

is its slope decomposition. The endomorphism object internal to the category of isocrystals $\underline{E}nd((D, \varphi))$ has as 0-graded piece

$$\bigoplus_{\lambda \in \mathbb{Q}} \underline{E}nd((D_\lambda, \varphi_\lambda)) \subseteq \underline{E}nd((D, \varphi)).$$

Analogously, if we fix $S \in \mathrm{Perf}_{K_0^\diamond}$ we have identifications of internal objects

$$\underline{E}nd(\mathcal{E}_S(D, \phi)) = \mathcal{E}_S(\underline{E}nd((D, \varphi))).$$

The right hand side is naturally graded and we have an injective map from the 0-graded piece

$$\bigoplus_{\lambda \in \mathbb{Q}} \underline{E}nd(\mathcal{E}_S(D_\lambda, \varphi_\lambda)) \subseteq \underline{E}nd(\mathcal{E}_S(D, \phi)).$$

Global sections of this later vector bundle are precisely the endomorphisms of $\mathcal{E}_S(D,\varphi)$ that respect the $\mathbb{Q}$-grading. Now, each term $\underline{E}nd(\mathcal{E}_S(D_\lambda,\varphi_\lambda))$ is an algebra whose underlying vector bundle is trivial. This last implies

$$H^0(\mathcal{X}_{FF,S},\bigoplus_{\lambda\in\mathbb{Q}}\underline{E}nd(\mathcal{E}_S(D_\lambda,\varphi_\lambda)))=Hom_{cont}(|S|,\bigoplus_{\lambda\in\mathbb{Q}}End_{\varphi-\mathrm{Mod}_{K_0}}(D_\lambda,\varphi_\lambda)).$$

Here the topology on $End_{\varphi-\mathrm{Mod}_{K_0}}(D_\lambda,\varphi_\lambda)$ is the one obtained from knowing that it is a finite dimensional $\mathbb{Q}_p$-vector space. Passing to units and recalling that

$$\bigoplus_{\lambda\in\mathbb{Q}}End_{\varphi-\mathrm{Mod}_{K_0}}(D_\lambda,\varphi_\lambda)=End_{\varphi-\mathrm{Mod}_{K_0}}(D,\varphi)$$

we get our desired map $\iota_{(D,\varphi)}:\underline{Aut(D,\varphi)}\to Aut(\mathcal{E}_S(D,\varphi))$ which identifies the left-hand group with the automorphisms of $\mathcal{E}_S(D,\varphi)$ that respect the $\mathbb{Q}$-grading.

Let us discuss the general case. Given an object $(V,\rho)\in\mathrm{Rep}_G(\mathbb{Q}_p)$ we get a natural map of algebraic groups $J_b\to Aut(\mathcal{F}_b(V,\rho))$. In particular, we get a continuous morphism $\psi_V:J_b(\mathbb{Q}_p)\to Aut(\mathcal{F}_b(V,\rho))(\mathbb{Q}_p)$. Given a continuous map $f:|S|\to J_b(\mathbb{Q}_p)$ we consider the composition $\psi_V\circ f$. This induces an automorphism of $\mathcal{E}_S(\mathcal{F}_b(V,\rho))$ that respects the $\mathbb{Q}$-grading, namely $\iota_{\mathcal{F}_b(V,\rho)}(\psi_V\circ f)$. If we are given a morphism $\pi:(V,\rho_V)\to(W,\rho_W)$ we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathcal{E}_S(\mathcal{F}_b(V,\rho_V)) \arrow[r, "\mathcal{E}(\mathcal{F}_b(\pi))"] \arrow[d, "\iota_{\mathcal{F}_b(V,\rho)}(\psi_V\circ f)"] & \mathcal{E}_S(\mathcal{F}_b(W,\rho_W)) \arrow[d, "\iota_{\mathcal{F}_b(V,\rho)}(\psi_W\circ f)"] \\
\mathcal{E}_S(\mathcal{F}_b(V,\rho_V)) \arrow[r, "\mathcal{E}(\mathcal{F}_b(\pi))"] & \mathcal{E}_S(\mathcal{F}_b(W,\rho_W))
\end{tikzcd}$$

This gives overall an automorphism of $\mathcal{E}_{b,S}$ that respects $\mathbb{Q}$-grading on each $\mathcal{E}_{b,S}(V,\rho)$. This constructs the map $\iota_b:\underline{J_b(\mathbb{Q}_p)}\to\mathcal{J}_b$ which clearly factors through J'_b. Conversely, assume we are given a map $m\in J'_b(S)$. For all $(V,\rho)\in\mathrm{Rep}_G(\mathbb{Q}_p)$ we obtain a continuous map

$$m_{(V,\rho)}:|S|\to Aut(\mathcal{F}_b(V,\rho))(\mathbb{Q}_p)\subseteq End_{\varphi-\mathrm{Mod}_{K_0}}(\mathcal{F}_b(V,\rho)).$$

Moreover, given an arrow $(V,\rho_V)\xrightarrow{\pi}(W,\rho_W)$ we obtain two maps

$$|S|\to Hom_{\varphi-\mathrm{Mod}_{K_0}}(\mathcal{F}_b(V,\rho_V),\mathcal{F}_b(W,\rho_W)).$$

One is given as the composition of $\mathcal{F}_b(\pi)$ with the family of endomorphisms $m_{(W,\rho_W)}$ and the other as the composition of $m_{(V,\rho_V)}$ with $\mathcal{F}_b(\pi)$ in the appropriate order. From the construction of $m_{(V,\rho)}$ these two endomorphisms coincide. We claim this determines a unique continuous map $|S|\to J_b(\mathbb{Q}_p)$. Indeed, $J_b(\mathbb{Q}_p)$ is the subgroup of $\prod_{(V,\rho)}Aut(\mathcal{F}_b(V,\rho))(\mathbb{Q}_p)$ that satisfies the commutativity constraints imposed by the arrows in $Rep_G(\mathbb{Q}_p)$. This gives a map $|S|\to J_b(\mathbb{Q}_p)$ which a priori is only continuous with respect to the weak topology making the maps $J_b(\mathbb{Q}_p)\to Aut(\mathcal{F}_b(V,\rho))(\mathbb{Q}_p)$ continuous. But if (V,ρ) is a faithful representation of G then the map of algebraic groups $J_b\to Aut(V,\rho)$ is a closed immersion. This gives that the weak topology on $J_b(\mathbb{Q}_p)$ is the p-adic topology.

Let us prove that the left action of $\underline{J_b(\mathbb{Q}_p)}$ on our moduli spaces through ι_b commutes, in an appropriate sense, with the Weil group action. The first thing we observe is that the group $\mathcal{J}_b$ itself comes equipped with Weil descent datum. Indeed, $\sigma^{op,*}\mathcal{J}_b$ is canonically identified with $\mathcal{J}_{\sigma(b)}$ and the isomorphism of bundles $\mathcal{E}_{\sigma(b)} \xrightarrow{\phi_b} \mathcal{E}_b$ induces a Weil descent datum

$$\tau_b : \mathcal{J}_b \to \sigma^{op,*}\mathcal{J}_b = \mathcal{J}_{\sigma(b)},$$

obtained from conjugating by ϕ_b. One readily verifies that the action map commutes with Weil descent datum, as in the diagram below.

$$\begin{tikzcd}
\mathcal{J}_b \times_{K_0} Gr_{K_0}(\mathcal{E}_b) \arrow[r, "m"] \arrow[d, "(\tau_b,\tau_b)"] & Gr_{K_0}(\mathcal{E}_b) \arrow[d, "\tau_b"] \\
\sigma^{op,*}\mathcal{J}_b \times_{K_0} \sigma^{op,*}Gr_{K_0}(\mathcal{E}_b) \arrow[r, "\sigma^{op,*}m"] & \sigma^{op,*}Gr_{K_0}(\mathcal{E}_b)
\end{tikzcd}$$

Indeed, both Weil descent data were defined by conjugating by ϕ_b. Mutatis mutandis the same applies to all the moduli spaces that appear in proposition 3.2.23 and the variants using $\hat{\sigma}$.

The constant group $\underline{J_b(\mathbb{Q}_p)}$ is defined over $\mathrm{Spd}(\mathbb{F}_p)$, this induces a canonical Weil descent datum on $(\underline{J_b(\mathbb{Q}_p)})_{K_0}$. Let us prove that the morphism

$$\iota_b : \underline{J_b(\mathbb{Q}_p)} \to \mathcal{J}_b$$

is compatible with Weil descent datum. Let $S \in \mathrm{Perf}_{\mathbb{F}_p}$, let $f : |S| \to J_b(\mathbb{Q}_p)$ be a continuous map and let $S^\sharp$ denote an untilt of S over K_0. For all $(V,\rho) \in Rep_G(\mathbb{Q}_p)$ we obtain from ι_b and f an automorphism of $\mathcal{E}_S(\mathcal{F}_b(V,\rho))$, and analogously we obtain from $\sigma^*\iota_b$ and f an automorphism of $\mathcal{E}_{S^\sigma}(\mathcal{F}_b(V,\rho)) = \mathcal{E}_S(\mathcal{F}_{\sigma(b)}(V,\rho))$. By abuse of notation we let $\sigma : J_b(\mathbb{Q}_p) \to J_{\sigma(b)}(\mathbb{Q}_p)$ denote the group isomorphism obtained from regarding $J_b(\mathbb{Q}_p)$ and $J_{\sigma(b)}(\mathbb{Q}_p)$ as subgroups of $G(K_0)$ and letting σ act on this later group. Consider the following diagram.

$$\begin{tikzcd}
(\underline{J_b(\mathbb{Q}_p)})_{K_0} \arrow[r, "\sigma"] \arrow[d, "\iota_b"] \arrow[dr, "\sigma^*\iota_b"] & (\underline{J_{\sigma(b)}(\mathbb{Q}_p)})_{K_0} \arrow[d, "\iota_{\sigma(b)}"] \\
\mathcal{J}_b \arrow[r, "\tau_b"] & \sigma^*\mathcal{J}_b = \mathcal{J}_{\sigma(b)}
\end{tikzcd}$$

To prove that ι_b is compatible with Weil descent datum one must verify that the lower triangle commutes. One way to do this is to verify that the upper triangle commutes and that the square commutes. Both commutativities are left to the verification of the careful reader. The commutativity of the upper triangle ultimately follows from the fact that if $h : K_0^n \to K_0^n$ is a K_0-linear automorphism given by a matrix (h_{ij}), then σ^*h is given by the matrix $(\sigma(h_{ij}))$. The commutativity of the square ultimately follows from the fact that if $g^{-1} \cdot b \cdot \sigma(g) = b$ then the identity $b^{-1} \cdot g \cdot b = \sigma(g)$ also holds.

From this we can conclude that the moduli spaces of proposition 3.2.23 come equipped with a K_0-linear (respectively $\breve{E}$-linear) left action by $\underline{J_b(\mathbb{Q}_p)}$ that commutes with the σ-linear (respectively $\hat{\sigma}$-linear) right action of $\underline{W_{K_0}}$ (respectively $\underline{W_{\breve{E}/E}}$). Indeed, the action map is compatible with Weil descent datum and since $\underline{J_b(\mathbb{Q}_p)}$ is a constant group defined over $\mathbb{F}_p$ the Weil group action on it is trivial.

3.2.9 Group functoriality

We start this subsection discussing a convention. As we have discussed above the space $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p})$ comes equipped naturally with a left action by $\underline{J_b(\mathbb{Q}_p)}$ and right actions by $\underline{G(\mathbb{Q}_p)}$ and $\underline{W_{\breve{E}/E}}$. We have also justified that these three actions commute. We may always replace the left $\underline{J_b(\mathbb{Q}_p)}$-action by a right $\underline{J_b(\mathbb{Q}_p)}$-action by defining $\alpha \cdot j := j^{-1} \cdot \alpha$. In this way we can say more succinctly that $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p})$ comes equipped with a right action by the group $\underline{G(\mathbb{Q}_p)} \times \underline{J_b(\mathbb{Q}_p)} \times \underline{W_{\breve{E}/E}}$. Moreover $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p})$ together with its right action by $\underline{G(\mathbb{Q}_p)} \times \underline{J_b(\mathbb{Q}_p)} \times \underline{W_{\breve{E}/E}}$ only depends on b through its associated element $[b] \in B(G)$.

In this section we briefly describe how this action behaves with respect to a morphism of algebraic groups. Fix such a morphism $\mathrm{f} : G \to H$ of reductive groups over $\mathbb{Q}_p$. Let $b_H = \mathrm{f}(b) \in H(L)$ and let $[\mu_H] = [\mathrm{f} \circ \mu]$. From the Tannakian definition of $\mathcal{E}_b := \mathcal{E} \circ \mathcal{F}_b$ and the identity $\mathcal{F}_{b_H} = \mathcal{F}_b \circ \mathrm{f}^*$ we get a canonical identification of H-torsors $\mathrm{f}_*\mathcal{E}_b = \mathcal{E}_{b_H}$ which defines a morphism

$$\mathrm{f}_{\infty,\infty} : \mathrm{Sht}_{G,b,[\mu],\infty} \to \mathrm{Sht}_{H,b_H,[\mu_H],\infty}$$

sending

$$[\alpha : G \dashrightarrow \mathcal{E}_b] \mapsto [\mathrm{f}_*\alpha : H \dashrightarrow \mathcal{E}_{b_H}].$$

Associated to b_H we can form $J_{b_H} = Aut^{\otimes}(\mathcal{F}_{b_H})$ and we get a morphism of algebraic groups $\mathrm{f} : J_b \to J_{b_H}$. We get commutative diagrams

$$\begin{tikzcd} \underline{J_b(\mathbb{Q}_p)} \arrow[r, "\iota_b"] \arrow[d, "\mathrm{f}"] & \mathcal{J}_b(\mathbb{Q}_p) \arrow[d, "\mathrm{f}"] \\ \underline{J_{b_H}(\mathbb{Q}_p)} \arrow[r, "\iota_{b_H}"] & \mathcal{J}_{b_H}(\mathbb{Q}_p) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Sht}_{G,b,[\mu],\infty} \arrow[r, "\tau_b"] \arrow[d, "\mathrm{f}_{\infty,\infty}"] & \hat{\sigma}^*\mathrm{Sht}_{G,b,[\mu],\infty} \arrow[d, "\hat{\sigma}^*\mathrm{f}_{\infty,\infty}"] \\ \mathrm{Sht}_{H,b_H,[\mu_H],\infty} \arrow[r, "\tau_{b_H}"] & \hat{\sigma}^*\mathrm{Sht}_{H,b_H,[\mu_H],\infty}. \end{tikzcd}$$

We conclude that the basechange of $\mathrm{f}_{\infty,\infty}$ to $\mathrm{Spd}(C_p, O_{C_p})$ is equivariant with respect to the $\underline{G(\mathbb{Q}_p)} \times \underline{J_b(\mathbb{Q}_p)} \times \underline{W_{\breve{E}/E}}$-action, where $\underline{G(\mathbb{Q}_p)} \times \underline{J_b(\mathbb{Q}_p)}$ acts on $\mathrm{Sht}_{H,b_H,[\mu_H],\infty}$ through the map

$$\mathrm{f} : \underline{G(\mathbb{Q}_p)} \times \underline{J_b(\mathbb{Q}_p)} \to \underline{H(\mathbb{Q}_p)} \times \underline{J_{b_H}(\mathbb{Q}_p)}$$

obtained from the map of algebraic groups $\mathrm{f} : G \times J_b \to H \times J_{b_H}$.

We may also impose a level structure $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ to get a family of morphisms

$$\mathrm{f}_{\mathcal{K},\mathrm{f}(\mathcal{K})} : \mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \to \mathrm{Sht}_{H,b_H,[\mu_H],\mathrm{f}(\mathcal{K})}.$$

This family of maps ranges over the compact subgroups of $G(\mathbb{Q}_p)$. Notice that even if $\mathcal{K}$ is open in $G(\mathbb{Q}_p)$, $\mathrm{f}(\mathcal{K})$ might not be open in $H(\mathbb{Q}_p)$. Each morphism in this family is $\underline{J_b(\mathbb{Q}_p)}$-equivariant and its basechange to $\mathrm{Spd}(C_p, O_{C_p})$ is $\underline{W_{\breve{E}/E}}$-equivariant.

3.3 The case of tori

3.3.1 Norm morphisms

In this section we study $\mathrm{Sht}_{G,b,[\mu],\infty} \times_{\breve{E}} \mathrm{Spd}(C_p, O_{C_p})$ together with its action by $\underline{G(\mathbb{Q}_p)} \times \underline{J_b(\mathbb{Q}_p)} \times \underline{W_{\breve{E}/E}}$ in the case in which G is a torus. We change our notation slightly and let $G = T$ for this case. We remark that this case was tackled by M. Chen in [7] and it was also thoroughly discussed in [14]. We recall the story in a different language.

By the work of Kottwitz we know that every element of $B(T)$ is basic and that the Kottwitz map $\kappa_T : B(T) \to \pi_1(T)_{\Gamma_{\mathbb{Q}_p}} = X_*(T_{\overline{\mathbb{Q}_p}})_{\Gamma_{\mathbb{Q}_p}}$ is a bijection. The sets $B(T, \mu)$ are singletons and are determined by the image of μ in $\pi_1(T)_{\Gamma_{\mathbb{Q}_p}}$.

Let us show that in the case of tori moduli spaces of p-adic shtukas are 0-dimensional.

Proposition 3.3.1. *If $b \in B(T, \mu)$ then all the maps in the following diagram are isomorphisms:*

$$\begin{tikzcd}
Gr^{adm,[\mu]}_{\breve{E}}(\mathcal{E}_b) \arrow[r] \arrow[dr] & Gr^{[\mu]}_{\breve{E}}(\mathcal{E}_b) \arrow[r] \arrow[d, "\pi_{BB}"] & Gr^{\leq[\mu]}_{\breve{E}}(\mathcal{E}_b) \arrow[d] \\
 & (\mathscr{F}l^{\omega_b}_{\breve{E},[-\mu]})^{\diamond} \arrow[r] & \mathrm{Spd}(\breve{E}, O_{\breve{E}})
\end{tikzcd}$$

Proof. The top and left arrows in the square are isomorphisms since μ is minuscule. Since T is a torus the only parabolic subgroup of T is itself, this gives $\mathscr{F}l^{\omega_b}_{\breve{E},[-\mu]} \cong T_{\breve{E}}/T_{\breve{E}} = \mathrm{Spec}(\breve{E})$.

Now, when $b \in B(T, \mu)$ the admissible locus $Gr^{adm,\leq[\mu]}_{\breve{E}}(\mathcal{E}_b)$ is non-empty and open within $Gr^{[\mu]}_{\breve{E}}(\mathcal{E}_b)$. Since $|\mathrm{Spd}(\breve{E}, O_{\breve{E}})| = \{*\}$ we must have $Gr^{adm,[\mu]}_{\breve{E}}(\mathcal{E}_b) = \mathrm{Spd}(\breve{E}, O_{\breve{E}})$. $\square$

On geometric points the situation is very simple, we have that the natural structure map $Gr^{adm,\leq\mu}_{C_p}(\mathcal{E}_b) \to \mathrm{Spd}(C_p, O_{C_p})$ is an isomorphism and

$$\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p \cong \underline{T(\mathbb{Q}_p)} \times \mathrm{Spd}(C_p, O_{C_p}),$$

since on geometric points every right $\underline{T(\mathbb{Q}_p)}$-torsor is trivial. It becomes more interesting when we compare the action of $\underline{J_b(\mathbb{Q}_p)}$ and $\underline{W_{\breve{E}/E}}$ to that of $\underline{T(\mathbb{Q}_p)}$. We begin by discussing the action of $\underline{J_b(\mathbb{Q}_p)}$.

Recall that if b is basic then J_b is an inner form of T, and that since T is commutative we must have $T = J_b$. More precisely we have a canonical inclusion $J_b(\mathbb{Q}_p) \subseteq T(K_0)$ that induces an isomorphism onto $T(\mathbb{Q}_p)$, we denote by j_b this identification.

Proposition 3.3.2. *The action of $\underline{T(\mathbb{Q}_p)}$ and $\underline{J_b(\mathbb{Q}_p)}$ are inverse to each other. In other words, if $S \in \mathrm{Perf}_{C_p}$, $f : |S| \to J_b(\mathbb{Q}_p)$ is a continuous map, and $\alpha \in \mathrm{Sht}_{T,b,[\mu],\infty} \times C_p$ then*

$$\alpha \cdot_{J_b(\mathbb{Q}_p)} f = \alpha \cdot_{T(\mathbb{Q}_p)} j_b(f^{-1}).$$

Before starting the proof of proposition 3.3.2 we recall the following lemma on Tannakian formalism:

Lemma 3.3.3. *Let X be a quasi-compact separated scheme over $\mathbb{Q}_p$, G an affine algebraic group over $\mathbb{Q}_p$ with center $Z(G)$ and let $\mathcal{T}_1$, $\mathcal{T}_2$ be two G-torsors over X, let $\mathcal{U}$ be a $\mathbb{Q}_p$-linear Tannakian category and let $\mathcal{F} : \mathrm{Rep}_T(\mathbb{Q}_p) \to \mathcal{U}$ denote an exact $\otimes$-functor.*

1. *There is a canonical injection $\iota_{\mathcal{F}} : Z(G)(\mathbb{Q}_p) \to \mathrm{Aut}^{\otimes}(\mathcal{F})$*
2. *There are canonical injections $\iota_i : Z(G)(\mathbb{Q}_p) \to \mathrm{Aut}_X(\mathcal{T}_i)$ for $i \in \{1, 2\}$.*
3. *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are isomorphic over X then the left action of $Z(G)(\mathbb{Q}_p)$ on $\mathrm{Isom}_U(\mathcal{T}_1, \mathcal{T}_2)$ through $\mathrm{Aut}_X(\mathcal{T}_1)$ coincides with the right action of $Z(G)(\mathbb{Q}_p)$ on $\mathrm{Isom}(\mathcal{T}_1, \mathcal{T}_2)$ through $\mathrm{Aut}_X(\mathcal{T}_2)$. That is, $\alpha \circ \iota_1(g) = \iota_2(g) \circ \alpha$ for every $g \in T(\mathbb{Q}_p)$ and $\alpha \in \mathrm{Isom}_X(\mathcal{T}_1, \mathcal{T}_2)$.*

Proof. The proof of the first claim and the second claim are very similar so we only prove the second. Let $\omega_{\mathcal{T}_1}$ and $\omega_{\mathcal{T}_2}$ denote the fiber functors associated to $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively. Consider the identity functor $Id : \mathrm{Rep}_G(\mathbb{Q}_p) \to \mathrm{Rep}_G(\mathbb{Q}_p)$, we have that $Z(G)(\mathbb{Q}_p) = \mathrm{Aut}^{\otimes}(Id) \subseteq \mathrm{Aut}^{\otimes}(\omega_{can,\mathbb{Q}_p}) = G(\mathbb{Q}_p)$. For any $g \in Z(G)(\mathbb{Q}_p)$ we let $\eta_g : Id \to Id$ denote the natural transformation that acts on (V, ρ) by $\rho(g)$. Notice that $\eta_g^{(V,\rho)} \in \mathrm{Hom}_{Rep_G}((V, \rho), (V, \rho))$ since g is central.

This gives the desired maps:

$$\iota_i : \mathrm{Aut}^{\otimes}(Id) \to \mathrm{Aut}^{\otimes}(\omega_{\mathcal{T}_i} \circ Id)$$

$$g \mapsto \omega_{\mathcal{T}_i}(\eta_g)$$

Let us prove the third claim, suppose now that $\alpha : \omega_{\mathcal{T}_1} \to \omega_{\mathcal{T}_2}$ is an isomorphism and let $g \in Z(G)(\mathbb{Q}_p)$. We have by definition $\iota_i(g) = \omega_{\mathcal{T}_i}(\eta_g)$. To prove the formula $\alpha \circ \iota_1(g) = \iota_2(g) \circ \alpha$ we must prove that the following diagram is commutative:

$$\begin{tikzcd}
\omega_{\mathcal{T}_1}(V, \rho) \arrow[r, "\alpha"] \arrow[d, "\omega_{\mathcal{T}_1}(\eta_g)"] & \omega_{\mathcal{T}_2}(V, \rho) \arrow[d, "\omega_{\mathcal{T}_2}(\eta_g)"] \\
\omega_{\mathcal{T}_1}(V, \rho) \arrow[r, "\alpha"] & \omega_{\mathcal{T}_2}(V, \rho)
\end{tikzcd}$$

But $\eta_g : (V, \rho) \to (V, \rho)$ is a morphism in $\mathrm{Rep}_G(\mathbb{Q}_p)$, so by definition of natural transformation the diagram must be commutative.

$\square$

Proof of proposition 3.3.2. We will justify the claim with the aid of the following commutative diagram which we explain below:

$$\begin{tikzcd}
\mathrm{Aut}_{\mathrm{X}_{FF,C_p}}(\mathcal{E}_e) & & T(\mathbb{Q}_p) \arrow[ll, "\iota_{\mathcal{E}_b}"'] \arrow[rr, "\iota_{\mathcal{E}_e}"] \arrow[dll, "\iota_{\mathcal{F}_e}"] \arrow[drr, "\iota_{\mathcal{F}_b}"] & & \mathrm{Aut}_{\mathrm{X}_{FF,C_p}}(\mathcal{E}_b) \\
J_e(\mathbb{Q}_p) \arrow[u] \arrow[drr, "\iota_e"] & & & & J_b(\mathbb{Q}_p) \arrow[u] \arrow[dll, "j_b"] \\
& & T(K_0) & &
\end{tikzcd}$$

Recall that $\mathcal{F}_b$ and $\mathcal{F}_e$ denote isocrystals with T-structure, that $J_b(\mathbb{Q}_p) = \mathrm{Aut}^{\otimes}(\mathcal{F}_b)$ and that $\mathcal{E}_b = \mathcal{E} \circ \mathcal{F}_b$. The triangles on the left and right of the diagram correspond to the triangles:

$$\begin{tikzcd}
\mathrm{Aut}^{\otimes}(Id) \arrow[rr] \arrow[dr] & & \mathrm{Aut}^{\otimes}(\mathcal{F}_b \circ Id) \arrow[dl] \\
& \mathrm{Aut}^{\otimes}(\mathcal{E} \circ \mathcal{F}_b \circ Id) &
\end{tikzcd}$$

In particular, the triangles on the first diagram are commutative. The bottom square corresponds to the concrete computation of $J_b(\mathbb{Q}_p)$ as a σ-centralizer that is

$$J_b(\mathbb{Q}_p) = \{g \in G(K_0) \mid g^{-1} b \sigma(g) = b\},$$

since T is abelian this is $T(K_0)^{\sigma = Id} = T(\mathbb{Q}_p)$. This implies that the maps $\iota_{\mathcal{F}_b}$ and $\iota_{\mathcal{F}_e}$ of lemma 3.3.3 are isomorphisms and we have that $j_b = \iota_{\mathcal{F}_b}^{-1}$.

By lemma 3.3.3, for all $\alpha \in \mathrm{Isom}_{\mathrm{X}_{FF,C_p} \setminus \infty}(\mathcal{E}_e, \mathcal{E}_b)$ and all $t \in T(\mathbb{Q}_p)$ we have $\iota_{\mathcal{F}_b}(t) \circ \alpha = \alpha \circ \iota_{\mathcal{F}_e}(t)$. We can compute the right action of $J_b(\mathbb{Q}_p)$ as follows:

$$\begin{aligned}
\alpha \cdot_{J_b(\mathbb{Q}_p)} j &= j^{-1} \circ \alpha \\
&= \iota_{\mathcal{F}_b}(j_b(j^{-1})) \circ \alpha \\
&= \alpha \circ \iota_{\mathcal{F}_e}(j_b(j^{-1})) \\
&= \alpha \cdot_{T(\mathbb{Q}_p)} j_b(j^{-1})
\end{aligned}$$

On the other hand,

$$\mathrm{Sht}_{T,b,[\mu],\infty}(C_p) \subseteq \mathrm{Isom}_{\mathrm{X}_{FF,C_p} \setminus \infty}(\mathcal{E}_e, \mathcal{E}_b),$$

and this inclusion is $T(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$-equivariant. Moreover, the natural map of sets

$$\mathrm{Sht}_{T,b,[\mu],\infty}(C_p) \to |\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p|$$

is bijective and the $\underline{J_b(\mathbb{Q}_p)}$-action is determined by the $J_b(\mathbb{Q}_p)$-action. This finishes the proof. $\square$

Let us study the Weil group action. In contrast to the actions of $J_b(\mathbb{Q}_p)$ and $T(\mathbb{Q}_p)$ the action of $W_{\breve{E}/E}$ on $\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p$ is not C_p-linear. In particular, we can only compare the actions of $W_{\breve{E}/E}$ and $T(\mathbb{Q}_p)$ on those invariants of $\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p$ that do not depend on the structure morphism to $\mathrm{Spd}(C_p, O_{C_p})$. In our case we compare the continuous actions on the topological space of connected components. As we have seen above this topological space is a topological right $T(\mathbb{Q}_p)$-torsor. Let $x \in \pi_0(\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p)$ and $\gamma \in W_{\breve{E}/E}$. We have

$$x \cdot_{W_{\breve{E}/E}} \gamma = x \cdot_{G(\mathbb{Q}_p)} g_{\gamma,x}$$

for a unique element $g_{\gamma,x} \in T(\mathbb{Q}_p)$. Since the actions of $W_{\breve{E}/E}$ and $T(\mathbb{Q}_p)$ commute we get a group homomorphism $g_{-,x} : W_{\breve{E}}^{op} \to T(\mathbb{Q}_p)$. Since $T(\mathbb{Q}_p)$ is commutative this morphism is independent of x. Moreover, the naive map of sets $\gamma \mapsto g_{\gamma,x}$ which would usually not be a group homomorphism is a group homomorphism again by the commutativity of $T(\mathbb{Q}_p)$. We denote this later group homomorphism by

$$m_{T,\mu} : W_{\breve{E}/E} \to T(\mathbb{Q}_p).$$

The following line of reasoning is taken from [44] lemma 1.22, which in turn is an elaboration of an argument in [34] page 413/41. Let $E \subseteq \overline{\mathbb{Q}_p}$ denote a finite field extension let $\{\mathrm{Tori}_{\mathbb{Q}_p}\}$ denote the category of tori defined over $\mathbb{Q}_p$. Recall the functor $X_*(-) : \{\mathrm{Tori}_{\mathbb{Q}_p}\} \to$ Sets given by the set of maps $\mathbb{G}_m \to T_{\overline{\mathbb{Q}_p}}$. Consider the subfunctor $X_*^E \subseteq X_*$ given by the subset of maps $\mathbb{G}_m \to T_{\overline{\mathbb{Q}_p}}$ whose field of definition is E. This functor is representable by $\mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m$ and comes equipped with a universal cocharacter $\mu_u \in X_*^E(\mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m)$. In other words, given a torus $T \in \{\mathrm{Tori}_{\mathbb{Q}_p}\}$ and $\mu \in X_*^E(T)$ there is a unique map $Nm_\mu : \mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m \to T$ of algebraic groups over $\mathbb{Q}_p$ such that $Nm_\mu \circ \mu_u = \mu$ in $X_*(T)$. The universal cocharacter can be expressed on E-points as follows:

$$E^\times \xrightarrow{e \mapsto e \otimes e} (E \otimes E)^\times.$$

Associated to μ_u there is a unique element of $[b_u] \in B(\mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m, \mu_u)$ since the Kottwitz map $\kappa : B(G) \to \pi_1(G)_{\Gamma_{\mathbb{Q}_p}}$ is bijective for tori. We fix a representative $b_u \in \mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m(\breve{\mathbb{Q}}_p)$ and abreviate by m_{E,μ_u} the map $m_{(\mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m,\mu_u)}$ previously constructed.

We can compute the $W_{\breve{E}/E}$-action on $|\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p|$ by reducing it to the universal case. Suppose we are given $\mu \in X_*^E(T)$ and $b \in T(K_0)$ with $[b] \in B(T, \mu)$, then automatically (b, μ) is admissible as in definition 3.2.13 and from the functoriality of the Kottwitz map we have that $[Nm_\mu(b_u)] = [b]$ in $B(T)$. We may replace b by $Nm_\mu(b_u)$ and we get a norm morphism

$$Nm_\mu : \mathrm{Sht}_{\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m),b_u,[\mu_u],\infty} \times C_p \to \mathrm{Sht}_{T,b,[\mu],\infty} \times C_p.$$

This map is $E^\times \times W_{\breve{E}/E}$-equivariant when the right space is endowed with the action induced from the map $Nm_\mu : \mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m)(\mathbb{Q}_p) = E^\times \to T(\mathbb{Q}_p)$. We can deduce the following.

Proposition 3.3.4. *Let the notation be as above, for all $T \in \{\mathrm{Tori}_{\mathbb{Q}_p}\}$ and $\mu \in X_*^E(T)$ we* *have*

$$m_{T,\mu} = Nm_\mu \circ m_{E,\mu_u}$$

as maps $W_{\breve{E}/E} \to T(\mathbb{Q}_p)$.

Proof. Fix $x \in \pi_0(\mathrm{Sht}_{\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m),b_u,[\mu_u],\infty} \times C_p)$ with image $y \in \pi_0(\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p)$ and $\gamma \in W_{\breve{E}/E}$. The equivariance of the norm map with respect to $E^\times$ and $W_{\breve{E}/E}$ allow us to compute:

$$\begin{aligned} y \cdot_{T(\mathbb{Q}_p)} m_{T,\mu}(\gamma) &= y \cdot_{W_{\breve{E}/E}} \gamma \\ &= Nm_\mu(x \cdot_{W_{\breve{E}/E}} \gamma) \\ &= Nm_\mu(x \cdot_{E^\times} m_{E,\mu_u}(\gamma)) \\ &= y \cdot_{T(\mathbb{Q}_p)} Nm_\mu(m_{E,\mu_u}(\gamma)) \end{aligned}$$

□

3.3.2 The Weil group action on the Lubin-Tate case

Our task now is to compute the action of $W_{\breve{E}/E}$ on $|\mathrm{Sht}_{\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m),b_u,[\mu_u],\infty} \times C_p|$. This is the only section in which it will pay off to let k be a finite field. Let $E \subseteq \overline{\mathbb{Q}_p}$ be a finite field extension of $\mathbb{Q}_p$, and fix a uniformizer $\pi \in E$. We let $F \subseteq E$ denote the maximal unramified extension, we let $h = [E : \mathbb{Q}_p]$ and we let $s = [F : \mathbb{Q}_p]$. Let $H_{LT,\pi}$ denote a Lubin-Tate formal group law with respect to π [37]. We may think of $H_{LT,\pi}$ as a p-divisible group defined over $\mathcal{O}_E$ and endowed with a strict $\mathcal{O}_E$-action ([13]). This means that the induced $\mathcal{O}_E$-action on $Lie(H_{LT})$ is the canonical one. As a p-divisible group H_{LT} has height h and dimension 1.

We let $\mathbb{M}_{LT} = M(H_{LT,\mathbb{F}_{p^s}})$ denote the covariant Dieudonné module over F obtained from Grothendieck-Messing theory [40]. We normalize the action of Frobenious on the covariant Dieudonné theory as in [6], [54], [52]. Let $\mathcal{M}_{LT}$ denote the Lie algebra of the universal vector extension of H_{LT} over $\mathcal{O}_E$. We have a canonical identification $\mathcal{M}_{LT}[\frac{1}{\pi}] = \mathbb{M}_{LT} \otimes_F E$, this allows us to endow $\mathbb{M}_{LT}$ with the usual one step filtration with $Fil^{-1}(\mathbb{M}_{LT} \otimes_F E) = \mathbb{M}_{LT} \otimes_F E$ and

$$Fil^{-1}(\mathbb{M}_{LT} \otimes_F E)/Fil^0(\mathbb{M}_{LT} \otimes_F E) = Lie(H_{LT})[\frac{1}{\pi}].$$

This data gives an object $D_{LT} = (\mathbb{M}_{LT}, \varphi_{LT}, Fil^\bullet(\mathbb{M}_{LT} \otimes_F E))$ in the category of weakly admissible filtered isocrystals. Moreover, due to our normalization of Frobenious action, the crystalline representation associated by Fontaine, $V_{cris}(D_{LT})$, gets identified on the nose with the rational Tate module of H. That is, $V_{cris}(D_{LT}) = T_p(H_{LT})[\frac{1}{p}]$ as Γ_E-representations, we let V_{LT} denote this representation.

The action of $\mathcal{O}_E$ on H_{LT} induces an action of $E^\times$ on D_{LT} and on V_{LT} respecting all structures, this way we may endow D_{LT} and V_{LT} with $\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m)$-structure if we reason as in [43] remark 3.4. Since $\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m)$ is a torus there is a unique cocharacter $\mu_{LT} \in X_*^E(\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m))$ defining the filtration on D_{LT}. We compute μ_{LT}.

We may think of $\mathbb{M}_{LT}$ as an $E \otimes_{\mathbb{Q}_p} F$ module endowed with $Id \otimes \sigma$-linear automorphism φ_{LT}. We get a decomposition

$$\mathbb{M}_{LT} = \bigoplus_{\iota: F \to E} (\mathbb{M}_{LT})_\iota$$

of E-vector spaces where F-acts on $(\mathbb{M}_{LT})_\iota$ through the embedding $\iota : F \to E$. Since φ_{LT} permutes these embeddings we get that each $(\mathbb{M}_{LT})_\iota$ has E-dimension 1. This in particular implies that $(\mathbb{M}_{LT})$ is a rank 1 free $E \otimes_{\mathbb{Q}_p} F$-module. We get a decomposition

$$\mathbb{M}_{LT} \otimes_F E = \bigoplus_{e \in Idem} (\mathbb{M}_{LT} \otimes_F E)_e$$

of $E \otimes_{\mathbb{Q}_p} E$-modules where e ranges over the idempotent elements of $E \otimes_{\mathbb{Q}_p} E$. The cocharacter μ_{LT} corresponds to a grading of $\mathbb{M}_{LT} \otimes_F E$ compatible with this decomposition. Moreover, $gr^{-1}(\mathbb{M}_{LT} \otimes_F E)$ maps isomorphically onto $Lie(H_{LT})[\frac{1}{\pi}]$. Let e_Δ denote the idempotent associated to the diagonal map $\Delta : E \otimes_{\mathbb{Q}_p} E \to E$. Since the action of $\mathcal{O}_E$ on $H_{LT,\pi}$ is strict the action of $E \otimes_{\mathbb{Q}_p} E$ on $Lie(H_{LT})[\frac{1}{\pi}]$ is through Δ (i.e. $(e_1 \otimes e_2) \cdot m = e_1 \cdot e_2 \cdot m$). We have that $gr^{-1}(\mathbb{M}_{LT} \otimes_F E) = (\mathbb{M}_{LT} \otimes_F E)_{e_\Delta}$ and consequently $gr^0(\mathbb{M}_{LT} \otimes_F E) = \bigoplus_{e \neq e_\Delta} (\mathbb{M}_{LT} \otimes_F E)_e$. The cocharacter $\mathbb{G}_{m,E} \to \mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m)_E$ that defines this grading is on E-valued points the following:

$$E^\times \xrightarrow{-1} E^\times \xrightarrow{e \mapsto e \otimes e} (E \otimes E)^\times$$

In other words, $\mu_{LT} = -\mu_u$. This information is already enough to compute the Weil group action on $\mathrm{Sht}_{\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m), b_u, [\mu_u], \infty} \times C_p$.

Consider the following identity and notice again the change of signs coming from remark 3.2.2

$$Gr_E^{adm,[\mu_u]}(\mathcal{E}_{\mathbb{M}_{LT}}) = \mathscr{F}l^{\omega_{b_u}}_{E,[\mu_{LT}]} = \mathrm{Spd}(E, O_E).$$

On this space, $\mathbb{L}$ is characterized by the crystalline representation it defines since this space consists of only one point. See remarks 3.2.3 and 3.2.5 and proposition 3.2.14. From the compatibility of Fontaine's functor with the Tate module we deduce that the crystalline representation associated to $\mathbb{L}$ is the left action of Γ_E on $T_p(H_{LT,\pi})[\frac{1}{p}]$.

After choosing a $E \otimes_{\mathbb{Q}_p} F$ basis for $\mathbb{M}_{LT}$ and letting b_u denote the action of φ_{LT} we get an isomorphism

$$Triv(\mathbb{L}) \times C_p \cong \mathrm{Sht}_{\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m), b_u, [\mu_u], \infty} \times C_p$$

where the space on the left denotes the moduli space of trivializations of $\mathbb{L}$. The space $Triv(\mathbb{L}) \times C_p$, being defined over $\mathrm{Spd}(E, O_E)$, comes equipped with a canonical Γ_E^{op}-action, but we emphasize that this action is not compatible with the Weil group action $W_{\breve{E}}^{op} \subseteq \Gamma_E^{op}$ on $\mathrm{Sht}_{\mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m), b_u, [\mu_u], \infty} \times C_p$ that we defined in section §2.7. Despite this, the canonical action on $Triv(\mathbb{L}) \times C_p$ will allow us to compute the $W_{\breve{E}/E}$-action we are interested in.

Let k denote an algebraically closed field extension of $\mathbb{F}_{p^s}$ and K_0 as on the notation section. The Weil group action on $Triv(\mathbb{L}) \times C_p$ that we are interested in comes from replacing the canonical Weil descent datum by the Weil descent datum τ induced from the automorphism

$$(\varphi_{LT}^s)^{-1} : \mathbb{M}_{LT} \to (Id \otimes \sigma^s)^* \mathbb{M}_{LT} = \mathbb{M}_{LT}.$$

Let $\gamma \in W_{\breve{E}/E}$ with $\gamma_{|K_0} = \sigma^{n \cdot s}$, and let

$$\Theta_{can}, \Theta_{Weil} : W_{\breve{E}/E}^{op} \to Aut(Triv(\mathbb{L}) \times C_p)$$

denote the action morphisms coming from the canonical and from the "φ_{LT}^s-modified" Weil descent data. Then $\Theta_{can}(\gamma)^{-1} \cdot \Theta_{Weil}(\gamma) = \tau^n$ with τ^n as in definition 3.2.20.

Now, recall that in the standard (or classical) normalization of covariant Dieudonné theory one defines the isocrystal structure $\psi_{LT} : \sigma^* \mathbb{M}_{LT} \to \mathbb{M}_{LT}$ by defining $\psi_{LT} = \mathbb{M}(\mathcal{V})$ where $\mathcal{V} : H_{LT}^{(p)} \to H_{LT}$ is the Verschiebung map. In the normalization we use we have by definition $\varphi_{LT} := \frac{\psi_{LT}}{p}$. Recall that $\psi_{LT} \circ \mathbb{M}(Frob_{H_{LT}}) = p$, in other words $\varphi_{LT} = \mathbb{M}(Frob_{H_{LT}})^{-1}$. This gives φ_{LT}^s coincides with $\mathbb{M}(Frob_{LT}^{-s})$. If we consider the multiplication map $[\pi] : H_{LT} \to H_{LT}$ restricted to $\mathrm{Spec}(\mathbb{F}_{p^s})$ we see from the definition of a Lubin-Tate formal group law that it agrees with the s-Frobenious automorphism of schemes. That is $Frob_{LT}^{-s}$ coincides with $\frac{1}{\pi}$ as quasi-isogenies. Overall this implies that the action of φ_{LT}^s on $\mathbb{M}_{LT}$ is multiplication by $\frac{1}{\pi} \otimes 1$, and consequently τ acts on $Triv(\mathbb{L})$ via multiplication by $\pi \in E^\times = \mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m)(\mathbb{Q}_p)$.

We claim now that $m_{\mu,E} = Art_E$ where Art_E denotes Artin's reciprocity map. Indeed, since the crystalline representation associated to $\mathbb{L}$ is the Lubin-Tate character, the action of Θ_{can} on $\pi_0(Triv(\mathbb{L}) \times C)$ when restricted to the inertia subgroup I_E is through the inverse of the Lubin-Tate character. Notice again the sign change, this was discussed on remark 3.2.7. This also gives the action of Θ_{Weil} since Θ_{can} and Θ_{Weil} agree on I_E. If $\hat{\sigma}_\pi$ denotes the unique lift of Frobenious on W_E^{ab} with $\hat{\sigma}_{\pi|E_\pi} = Id$ with E_π the Lubin-Tate extension associated to π, we see that $\Theta_{can}(\hat{\sigma}_\pi)$ acts trivially on $\pi_0(Triv(\mathbb{L}))$. This gives that $\Theta_{Weil}(\hat{\sigma}_\pi)$ acts on $\pi_0(Triv(\mathbb{L}))$ by τ which is multiplication by π. Specifying the action of I_E and of $\hat{\sigma}_\pi$ is one way of characterizing Artin's reciprocity map Art_E.

The following statement summarizes the results discussed on this section, for this statement we let $k = \overline{k}$:

Theorem 3.3.5. *(Compare with [7] 4.1) Let T be a torus over $\mathbb{Q}_p$, $b \in T(K_0)$, $\mu \in X_*(T)$ with $[b] \in B(T, \mu)$. Let $E \subseteq C_p$ be the field of definition of μ, let $\mathrm{Art}_E : W_E \to (\Gamma_E)^{ab} \to E^\times$ denote Artin's reciprocity character of local class field theory, let $Nm_\mu : \mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m) \to T$ be the unique map with $Nm_\mu \circ \mu_u$ as discussed above and let $Art_{\breve{E}/E}$ denote the composition $Art_{\breve{E}/E} : W_{\breve{E}/E} \to W_E \xrightarrow{Art_E} E^\times$, where the map $W_{\breve{E}/E} \to W_E$ is the one induced by the inclusion of fields $E \subseteq \breve{E} \subseteq C_p$. Then the following hold:*

1. *$\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p$ is a trivial right $\underline{T(\mathbb{Q}_p)}$-torsor over $\mathrm{Spd}(C_p, O_{C_p})$.*

2. *If $s \in \pi_0(\mathrm{Sht}_{T,b,[\mu],\infty} \times C_p)$ and $(g, j, \gamma) \in T(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_{\breve{E}/E}$ then*

$$s \cdot (g, j, \gamma) = s \cdot (g \cdot j_b(j^{-1}) \cdot (Nm_\mu \circ Art_{\breve{E}/E}(\gamma)))$$

where $j_b : J_b(\mathbb{Q}_p) \to T(\mathbb{Q}_p)$ is the isomorphism specified by regarding $J_b(\mathbb{Q}_p)$ as a subgroup of $T(K_0)$.

Since we have a full description of the Galois action we can easily compute from theorem 3.3.5 the connected components of $\mathrm{Sht}_{T,b,[\mu],\infty}$ as a space over $\mathrm{Spd}(\breve{E}, O_{\breve{E}})$. The computation is easier to explain with the following lemma whose proof we leave to the reader:

Lemma 3.3.6. *Let $\mathcal{K}$ be a locally profinite group, let L a p-adic field with Galois group Γ_L and $\mathbb{L}_{\mathcal{K}}$ a pro-étale $\underline{\mathcal{K}}$-torsor over $\mathrm{Spd}(L, O_L)$. Define $Triv(\mathbb{L}_{\mathcal{K}})$ as the moduli of trivializations of $\mathbb{L}_{\mathcal{K}}$. Then:*

1. *If C is the p-adic completion of an algebraic closure of L, then the choice of a map $\alpha : \mathrm{Spd}(C, O_C) \to Triv(\mathbb{L}_K)$ determines a group homomorphism $\rho_\alpha : \Gamma_L^{op} \to \mathcal{K}$.*

2. *For any $k \in \mathcal{K}$ we have $\rho_{\alpha \cdot k} = k^{-1} \cdot \rho_\alpha \cdot k$.*

3. *The action of $\mathcal{K}$ on $\pi_0(Triv(\mathbb{L}_{\mathcal{K}}))$ is transitive.*

4. *If $\pi_0(\alpha)$ denotes the unique connected component to which $|\alpha|$ maps to, then the stabilizer subgroup is given by the formula $\mathcal{K}_{\pi_0(\alpha)} = \rho_\alpha(\Gamma_L^{op})$.*

Proposition 3.3.7. *Let $\mathcal{K} \subseteq T(\mathbb{Q}_p)$ denote the largest compact subgroup, the following statements hold.*

1. *$\pi_0(\mathrm{Sht}_{T,b,[\mu],\infty})$ is a free right $T(\mathbb{Q}_p)/Nm_\mu(Art_{\breve{E}/E}(\Gamma_{\breve{E}}))$-torsor.*

2. *$\pi_0(\mathrm{Sht}_{T,b,[\mu],\mathcal{K}}) = \pi_0(\mathrm{Sht}_{T,b,[\mu],\mathcal{K}} \times C_p)$ and it is a free right $T(\mathbb{Q}_p)/\mathcal{K}$-torsor.*

Proof. The first statement follow directly from lemma 3.3.6 and theorem 3.3.5. The second statement follows from the fact that the action of $\Gamma_{\breve{E}}$ is continuous so the action of this compact group factors through the maximal compact subgroup. $\square$

3.4 On the unramified case.

For this section $k = \overline{k}$. The purpose of this section is to compute $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times C_p)$ together with its right action by $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_{\breve{E}/E}$-action under the assumption that G is an unramified reductive group and that (b, μ) is HN-irreducible (See definition 3.2.15). We recall that in this case the reflex field is of the form $E = \mathbb{Q}_{p^s}$ for some $s \in \mathbb{N}$ and consequently $\breve{E} = K_0$. Nevertheless, with the notation we have chosen, $W_{\breve{E}/E}$ is the subgroup of W_{K_0} of those automorphisms of C_p that lift a power of $\sigma^s : K_0 \to K_0$.

3.4.1 Connected components of affine Deligne Lusztig Varieties

As it turns out, the connected components of moduli spaces of p-adic shtukas can be computed from knowledge about the connected components of affine Deligne-Lusztig varieties. In this section we recall the relation. Recall that if G is an unramified group then there is a connected reductive group over $\mathbb{Z}_p$ whose generic fiber is isomorphic to G. Let us fix such a model and by abuse of notation denote it by G. We let $\mathcal{K} = G(\mathbb{Z}_p)$ and we let $\breve{\mathcal{K}} = G(O_{K_0})$. Since we are assuming $k = \overline{k}$, the group G_{K_0} is split over K_0 and we have by the Cartan decomposition a bijection

$$\breve{\mathcal{K}}\backslash G(K_0)/\breve{\mathcal{K}} = X_*(T_{\overline{\mathbb{Q}_p}})$$

given by

$$\mu \mapsto p^\mu := \mu(p) \in T(K_0).$$

We may construct a map $\kappa_G : G(K_0) \to \pi_1(G)_{\Gamma_{\mathbb{Q}_p}}$. Given an element $b \in G(K_0)$ there is a unique $\mu' \in X_*(T_{\overline{\mathbb{Q}_p}})$ with $b \in \breve{\mathcal{K}}\backslash p^{\mu'}/\breve{\mathcal{K}}$. Then $\kappa_G(b)$ is defined to be $[\mu']$, the induced class of μ' in $\pi_1(G)_{\Gamma_{\mathbb{Q}_p}}$. This map is a group homomorphism that is well-defined on σ-conjugacy classes. Moreover, the map constructed in this way descends to the Kottwitz map $\kappa_G : B(G) \to \pi_1(G)_{\Gamma_{\mathbb{Q}_p}}$ that we discussed on section §2.3.

Recall that associated to a pair (b, μ) one can associate an affine Deligne Lusztig variety $X^{\leq\mu}_G(b)$. This is a perfect scheme (See [5]) over $\mathrm{Spec}(k)$ whose k-valued points can be described as:

$$X^{\leq\mu}_G(b)(k) = \left\{ g\cdot\breve{\mathcal{K}} \in G(K_0)/\breve{\mathcal{K}} \mid g^{-1}\cdot b\cdot\sigma(g) \in \breve{\mathcal{K}}\backslash p^{\mu'}/\breve{\mathcal{K}} \text{ with } \mu' \leq \mu \right\}$$

In [9], [41] [22], the problem of determining connected components of affine Deligne Lusztig varieties is thoroughly discussed. Although the description in full generality is complicated, in our situation (G reductive and $\mathcal{K}$ hyperspecial) the problem is completely settled. In the references provided above, the connected components are described in three steps. The first step is to pass to the case of a simple adjoint group and it is done as follows:

Theorem 3.4.1. *(See [9] 2.4.2) Let G^{ad} denote the adjoint quotient of G, then there are natural maps w_G and $w_{G^{ad}}$ and elements $c_{b,\mu} \in \pi_1(G)$ ($c_{b_{ad},\mu_{ad}} \in \pi_1(G^{ad})$ respectively) well-defined up to multiplication by $\pi_1(G)^{\Gamma_{\mathbb{Q}_p}}$ (respectively $\pi_1(G^{ad})^{\Gamma_{\mathbb{Q}_p}}$) making the following diagram commutative and Cartesian:*

$$\begin{tikzcd}
X^{\leq\mu}_G(b) \arrow[r] \arrow[d, "w_G"] & X^{\leq\mu_{ad}}_{G^{ad}}(b_{ad}) \arrow[d, "w_{G^{ad}}"] \\
\underline{c_{b,\mu}\pi_1(G)^{\Gamma_{\mathbb{Q}_p}}} \times \mathrm{Spec}(k) \arrow[r] & \underline{c_{b_{ad},\mu_{ad}}\pi_1(G^{ad})^{\Gamma_{\mathbb{Q}_p}}} \times \mathrm{Spec}(k)
\end{tikzcd}$$

In the statement above the two sets that appear on the lower horizontal arrow should be interpreted as discrete topological groups so that the product is a disjoint union of copies of $\mathrm{Spec}(k)$. Once one reduces the problem to the adjoint case, one can further simplify to the simple adjoint case by observing that if $G = G_1 \times G_2$ then we get a decomposition

$$X^{\leq \mu}_G(b) = X^{\leq \mu_1}_{G_1}(b_1) \times_k X^{\leq \mu_2}_{G_2}(b_2).$$

This is how the first step is completed in the references.

The second step in the strategy is to reduce the general simple adjoint group case to the case in which (b, μ) is HN-indecomposable. In this work we only consider the case in which (b, μ) is already HN-irreducible which is a stronger condition to being indecomposable. For this reason we do not review this step.

The third and final step is the determination of $\pi_0(X^{\leq \mu}_G(b))$ when G is simple adjoint and (b, μ) is HN-irreducible or when it is HN-indecomposable but not HN-irreducible. Again, we only review the HN-irreducible case.

Theorem 3.4.2. *([41] 1.1, [9] 1.1, [22] 8.1) If (b, μ) is HN-irreducible and $G = G^{ad}$ is simple and adjoint then $w_G : \pi_0(X^{\leq \mu}_G(b)) \to c_{b,\mu}\pi_1(G^{ad})^{\Gamma_{\mathbb{Q}_p}}$ is a bijection.*

In what follows we rephrase these result in a form that will be more useful for our purposes. For this let G^{der} denote the derived subgroup of G, let $G^{ab} := G/G^{der}$ the maximal abelian quotient and denote by $det : G \to G/G^{der}$ the quotient map. We will often refer to the quotient map $G \to G^{ab}$ as the determinant map.

Corollary 3.4.3. *If G^{der} is simply connected the natural map $det : X^{\leq \mu}_G(b) \to X^{\leq \mu_{ab}}_{G^{ab}}(b_{ab})$ induced from $det : G \to G^{ab}$ gives a bijection of connected components $\pi_0(X^{\leq \mu}_G(b)) \cong \pi_0(X^{\leq \mu_{ab}}_{G^{ab}}(b_{ab}))$ whenever (b, μ) is HN-irreducible.*

Remark 3.4.4. *Since $X^{\leq \mu_{ab}}_{G^{ab}}(b_{ab})$ is a disjoint union of copies of* $\mathrm{Spec}(k)$ *and* $\mathrm{Spec}(k)$ *is algebraically closed, we could say instead that the map $X^{\leq \mu}_G(b) \to X^{\leq \mu_{ab}}_{G^{ab}}(b_{ab})$ has geometrically connected fibers.*

Proof. For the convenience of the reader we provide an easy argument using theorems 3.4.2 and 3.4.1. A pair (b, μ) is HN-irreducible if and only if for every $\mathbb{Q}_p$-simple factor G_i of G^{ad} with projection map $\pi_i : G \to G_i$ the pair $(b_i, \mu_i) := (\pi_i(b), \pi_i \circ \mu)$ is HN-irreducible. Indeed, the coefficient of $\mu^{dom} - \nu_b^{dom}$ associated to a positive root can be computed on the simple factors of the adjoint quotient. From theorem 3.4.1 we get a Cartesian diagram:

$$\begin{tikzcd}
\pi_0(X^{\leq \mu}_G(b)) \arrow[r] \arrow[d, "w_G"] & \pi_0(X^{\leq \mu_1}_{G_1}(b_1)) \times \cdots \times \pi_0(X^{\leq \mu_n}_{G_n}(b_n)) \arrow[d, "w_{G_i}"] \\
c_{b,\mu}\pi_1(G)^{\Gamma_{\mathbb{Q}_p}} \arrow[r] & c_{b_1,\mu_1}\pi_1(G_1)^{\Gamma_{\mathbb{Q}_p}} \times \cdots \times c_{b_n,\mu_n}\pi_1(G_n)^{\Gamma_{\mathbb{Q}_p}}
\end{tikzcd}$$

The vertical right hand map is a bijection by theorem 3.4.2 which implies the vertical left hand map is also a bijection by theorem 3.4.1.

The result follows from showing that in the commutative diagram below the bottom horizontal arrow and the vertical right hand arrow are both bijective.

$$\begin{tikzcd}
\pi_0(X_G^{\leq\mu}(b)) \arrow[r] \arrow[d, "w_G"] & \pi_0(X_{G^{ab}}^{\leq\mu_{ab}}(b_{ab})) \arrow[d, "w_{G^{ab}}"] \\
c_{b,\mu}\pi_1(G)^{\Gamma_{\mathbb{Q}_p}} \arrow[r] & c_{b_{ab},\mu_{ab}}\pi_1(G^{ab})^{\Gamma_{\mathbb{Q}_p}}
\end{tikzcd}$$

Since G^{der} is simply connected we have a $\Gamma_{\mathbb{Q}_p}$-equivariant identification $\pi_1(G) \to \pi_1(G^{ab})$ so the bottom map is easily seen to be a bijection. Moreover, the adjoint quotient of G^{ab} is $\{e\}$ and theorem 3.4.1 says that $w_{G^{ab}} : X_{G^{ab}}^{\leq\mu_{ab}}(b_{ab}) \to c_{b_{ab},\mu_{ab}}\pi_1(G^{ab})^{\Gamma_{\mathbb{Q}_p}}$ is an isomorphism in this case.

□

Theorem 2 explains the role that affine Deligne-Lusztig varieties will play in our computation. Let us recall it in the notation of chapter 3.

Theorem 3.4.5. *Let G be an unramified reductive group over $\mathbb{Q}_p$, μ a conjugacy class of geometric cocharacters and $[b] \in B(G,\mu)$.*

a) There is a continuous and $J_b(\mathbb{Q}_p)$-equivariant specialization map

$$\mathrm{Sp} : |\mathrm{Sht}_{G,b,[\mu],\infty} \times C_p| \to |X_G^{\leq\mu}(b)|.$$

b) The specialization map induces a bijection of connected components

$$\pi_0(\mathrm{Sp}) : \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times C_p) \xrightarrow{\cong} \pi_0(X_G^{\leq\mu}(b)).$$

3.4.2 The simply connected case

In this subsection we compute $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty})$ under the assumption that G^{der} is simply connected.

Proposition 3.4.6. *Suppose that G is as above. The determinant map induces a surjective map of locally spatial diamonds*

$$det : \mathrm{Sht}_{G,b,[\mu],\infty} \to \mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty}$$

Proof. We may verify surjectivity after basechanging to an algebraic closure. Moreover, we can choose a section $s : \mathrm{Spa}(C,C^+) \to Gr_{K_0}^{adm,\leq[\mu]}(\mathcal{E}_b)$ and consider the following commutative diagram.

$$\begin{tikzcd}
\underline{G(\mathbb{Q}_p)} \times \mathrm{Spa}(C,C^+) \arrow[rr] \arrow[d, "\cong"] & & \underline{G^{ab}(\mathbb{Q}_p)} \times \mathrm{Spa}(C,C^+) \arrow[d, "\cong"] \\
\mathrm{Sht}_{G,b,[\mu],\infty} \times_{Gr^{\leq[\mu]}_{K_0}(\mathcal{E}_b)} \mathrm{Spa}(C,C^+) \arrow[rr] \arrow[d] & & \mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty} \times_{K_0} \mathrm{Spa}(C,C^+) \\
\mathrm{Sht}_{G,b,[\mu],\infty} \times_{K_0} \mathrm{Spa}(C,C^+) \arrow[urr] & &
\end{tikzcd}$$

We can consequently reduce to the surjectivity of $\underline{G(\mathbb{Q}_p)} \to \underline{G^{ab}(\mathbb{Q}_p)}$. That is, we must prove that can lift continuous maps $f \in C^0(|\mathrm{Spa}(R,R^+)|, G^{ab}(\mathbb{Q}_p))$ to a continuous map $\tilde{f} \in C^0(|\mathrm{Spa}(R,R^+)|, G(\mathbb{Q}_p))$. The key point is, of course, that since G^{der} is simply connected by Kneser's theorem [31] the map of groups $G(\mathbb{Q}_p) \to G^{ab}(\mathbb{Q}_p)$ is surjective.

Now, let $Z(G)$ denotes the center of G. We get a strict map of topological abelian groups $Z(G)[\mathbb{Q}_p] \to G^{ab}(\mathbb{Q}_p)$ with finite kernel and cokernel. $Im(Z(G)[\mathbb{Q}_p])$ is an open subgroups and there is a finite number of elements $g_1, \ldots, g_n \in G(\mathbb{Q}_p)$ with $\cup_{g_i} g_i \cdot Im(Z(G)[\mathbb{Q}_p]) = G^{ab}(\mathbb{Q}_p)$. The map $\cup_{g_i} g_i \cdot \underline{Z(G)[\mathbb{Q}_p])} \to \underline{G^{ab}(\mathbb{Q}_p)}$ is surjective and factors through $\underline{G(\mathbb{Q}_p)}$ which finishes the proof. $\square$

Lemma 3.4.7. *Let G be as above (unramified and such that $G^{der} = G^{sc}$). Let $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ be a hyperspecial subgroup. Suppose (b,μ) is HN-irreducible, then*

$$det : \mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \to \mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],det(\mathcal{K})}$$

has geometrically connected fibers.

Proof. Since G splits over an unramified extension, we can construct an exact sequence

$$e \to \mathcal{G}^{der} \to \mathcal{G} \to \mathcal{G}^{ab} \to e$$

of reductive groups over $\mathbb{Z}_p$. Indeed, this evident for split groups and we may use étale descent from $\mathrm{Spec}(\mathbb{Z}_{p^s})$ to $\mathrm{Spec}(\mathbb{Z}_p)$ in the general case. An application of Lang's theorem proves that $det(\mathcal{K}) = \mathcal{G}^{ab}(\mathbb{Z}_p)$ which is the maximal bounded subgroup of G^{ab}. By functoriality our results on chapter 1 and 2 we have a commutative diagram of specialization maps:

$$\begin{tikzcd}
\mid \mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times C_p \mid \arrow[r, "det"] \arrow[d, "Sp_{\mathcal{G}}"] & \mid \mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],det(\mathcal{K})} \times C_p \mid \arrow[d, "Sp_{\mathcal{G}^{ab}}"] \\
\mid X^{\leq\mu}_G(b) \mid \arrow[r, "det"] & \mid X^{\leq\mu_{ab}}_{G^{ab}}(b_{ab}) \mid
\end{tikzcd}$$

The vertical maps give bijections of connected components by theorem 3.4.5 and the lower horizontal map induces a bijection of connected components by corollary 3.4.3. $\square$

The following proposition is a particular case of an unpublished result of Hansen and Weinstein that follows from the work done in [19]. We provide an alternative proof that follows the steps of the analogous statement in [8] Lemme 6.1.3.

Proposition 3.4.8. *Let G be as above and let (b,μ) be HN-irreducible. Then $Gr^{adm,\leq[\mu]}_{K_0}(\mathcal{E}_b)$ is geometrically connected over* $\mathrm{Spd}(K_0, O_{K_0})$.

Proof. Let $\mathrm{Spa}(C, O_C) \to \mathrm{Spd}(K_0, O_{K_0})$ be a map with C a non-Archimedean algebraically closed field, $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ a hyperspecial subgroup, and let $\mathcal{M}$ denote a connected component of $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathrm{Spd}(C, O_C)$. We consider the restriction of the period morphism $\pi_{GM,\mathcal{K},C} : \mathcal{M} \to Gr^{adm,\leq[\mu]}_C(\mathcal{E}_b)$. By lemma 3.4.7, $\mathcal{M}$ is an open subdiamond of $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathrm{Spd}(C, O_C)$ and by étaleness of $\pi_{GM,\mathcal{K},C}$ the set $U := \pi_{GM,\mathcal{K},C}(\mathcal{M})$ is a connected open subset of $Gr^{adm,\leq[\mu]}_C(\mathcal{E}_b)$. We claim, and prove below, that this open subset doesn't depend on the choice of $\mathcal{M}$. This already implies $Gr^{adm,\leq[\mu]}_C(\mathcal{E}_b) = \pi_{GM,\mathcal{K},C}(\mathcal{M})$ and in particular that it is connected.

Let us prove the claim, for this we take a connected component $\mathcal{M}_\infty$ of $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C, O_C)$ that maps to $\mathcal{M}$. Notice that $\pi_{\infty,\mathcal{K}}(\mathcal{M}_\infty) = \mathcal{M}$ since the groups $\mathcal{K}' \subseteq \mathcal{K}$ of finite index are cofinal and for those the transition maps

$$\mathrm{Sht}_{G,b,[\mu],\mathcal{K}'} \times \mathrm{Spd}(C, O_C) \to \mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathrm{Spd}(C, O_C)$$

are finite étale and surjective so that on topological level the transition maps are open and closed. This also implies $U = \pi_{GM}(\mathcal{M}_\infty)$.

By lemma 3.4.7 $\pi_0(\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathrm{Spd}(C, O_C)) \to \pi_0(\mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],det(\mathcal{K})})$ is a bijection. Let $\mathcal{M}'$ denote some other connected component, and let z and z' denote the elements defined by $\mathcal{M}$ and $\mathcal{M}'$ in $\pi_0(\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathrm{Spd}(C, O_C))$. Now, $G^{ab}(\mathbb{Q}_p)$ acts transitively on

$$\pi_0(\mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty} \times \mathrm{Spd}(C, O_C))$$

and consequently $G^{ab}(\mathbb{Q}_p)/det(\mathcal{K})$ acts transitively on

$$\pi_0(\mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],det(\mathcal{K})} \times \mathrm{Spd}(C, O_C)).$$

This allow us to find an element $g \in G(\mathbb{Q}_p)$ with $det(z) \cdot det(g) = det(z')$. Let $x : \mathrm{Spd}(C, C^+) \to U$ be a geometric point and let $\overline{x} : \mathrm{Spd}(C, C^+) \to \mathcal{M}_\infty$ be a lift of x. Consider $\overline{x} \cdot g$. On one hand it is a lift of x, and on the other hand its projection to $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathrm{Spd}(C, O_C)$ lands on $\mathcal{M}'$. Indeed, we have a commutative diagram:

$$\begin{tikzcd}
\mathrm{Sht}_{G,b,[\mu],\infty}(C, C^+) \arrow[r, "det"] \arrow[d, "\pi_{\infty,\mathcal{K}}"] & \mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty}(C, C^+) \arrow[d, "\pi_{\infty,det(\mathcal{K})}"] \\
\mathrm{Sht}_{G,b,[\mu],\mathcal{K}}(C, C^+) \arrow[r, "det"] & \mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],det(\mathcal{K})}(C, C^+)
\end{tikzcd}$$

We have:

$$\begin{aligned}
det \circ \pi_{\infty,\mathcal{K}}(\overline{x} \cdot g) &= \pi_{\infty,det(\mathcal{K})} \circ det(\overline{x} \cdot g) \\
&= \pi_{\infty,det(\mathcal{K})}[det(\overline{x}) \cdot det(g)] \\
&= \pi_{\infty,det(\mathcal{K})} \circ det(\overline{x}) \cdot det(g)
\end{aligned}$$

This map lands on $det(z)\cdot det(g)$ which is $det(z')$. This implies that $\pi_{\infty,\mathcal{K}}(\overline{x}\cdot g)$ is a geometric point on $\mathcal{M}'$.

This proves that any topological point of U also comes from a point in $\mathcal{M}'$, and that $\pi_{GM}(\mathcal{M}) \subseteq \pi_{GM}(\mathcal{M}')$. Since the roles of $\mathcal{M}$ and $\mathcal{M}'$ in the proof can be reversed the converse also holds. ☐

Lemma 3.4.9. *Let $\mathcal{K}$ be a hyperspecial subgroup of $G(\mathbb{Q}_p)$ and let $\mathcal{K}^{der} = \mathcal{K} \cap G^{der}(\mathbb{Q}_p)$. Let $m \in \pi_0(\mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty} \times \mathrm{Spd}(C, O_C))$ and let X_m denote the space defined by the following Cartesian diagram:*

$$\begin{tikzcd}
X_m \arrow[r] \arrow[d] & \mathrm{Spd}(C, O_C) \arrow[d, "m"] \\
\mathrm{Sht}_{G,b,[\mu],\infty} \times C \arrow[r, "det"] & \mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty} \times C.
\end{tikzcd}$$

Then $\mathcal{K}^{der}$ acts transitively on $\pi_0(X_m)$.

Proof. Since $\mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty} \times \mathrm{Spd}(C, O_C)$ is 0-dimensional, the space X_m is the collection of connected components of $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C, O_C)$ that map to m. Let $x, y \in \pi_0(X_m)$, using lemma 3.4.7 we see that $\pi_{\infty,\mathcal{K}}(x) = \pi_{\infty,\mathcal{K}}(y)$, we let $\mathcal{M}$ denote this connected component. Since $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C, O_C)$ is a $\underline{\mathcal{K}}$-torsor over $\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \times \mathrm{Spd}(C, O_C)$, $\mathcal{K}$ acts transitively on the set of connected components of $\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C, O_C)$ over $\mathcal{M}$. In particular, there is an element $g \in \mathcal{K}$ with $x \cdot g = y$. Since $det(x) = det(y)$ we must have that $m \cdot det(g) = m$, but the action of $G^{ab}(\mathbb{Q}_p)$ on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C, O_C))$ is simple so $det(g) = e$ and $g \in G^{der}(\mathbb{Q}_p)$ as we wanted to show. ☐

We can now describe connected components at infinite level.

Theorem 3.4.10. *Suppose G is an unramified group over $\mathbb{Q}_p$, suppose that G^{der} is simply connected and suppose that (b, μ) is HN-irreducible, then the determinant map*

$$det_{\infty,\infty} : \mathrm{Sht}_{G,b,[\mu],\infty} \to \mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty}$$

has connected geometric fibers.

Proof. Since $\mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty} \times \mathrm{Spd}(C_p, O_{C_p})$ is isomorphic to $\underline{G^{ab}}(\mathbb{Q}_p) \times \mathrm{Spd}(C_p, O_{C_p})$, we may prove instead that the determinant map induces a bijection

$$\pi_0(det) : \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p})) \to \pi_0(\mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty} \times \mathrm{Spd}(C_p, O_{C_p})).$$

Indeed, we may use [51] 16.2 which says that cohomology of a locally spatial diamond is invariant under the change of geometric point. In particular, this applies to the set of connected components since it is a cohomological invariant.

Let $x \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p}))$. Given K a finite extension of K_0 we let x_K denote the image of x on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K, O_K))$ and let $f : \mathrm{Spd}(K, O_K) \to Gr^{adm,\leq[\mu]}(\mathcal{E}_b)$

be a point whose associated crystalline representation is as in corollary 3.2.18. Let $S_f := Triv(f^*(\mathbb{L}))$ the geometric realization of $f^*\mathbb{L}$. This space is also the fiber over f of the infinite level Grothendieck-Messing period map. Let $s \in \pi_0(S_f)$ be an element mapping to x_K. In summary we have taken a commutative diagram as follows:

$$\begin{tikzcd} * \arrow[r, "x"] \arrow[d, "s"] \arrow[dr, "x_K"] & \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p})) \arrow[d] \\ \pi_0(S_f) \arrow[r, "f"] & \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K, O_K)) \end{tikzcd}$$

We let G^{der}_x (respectively $G^{der}_{x_K}$ and G^{der}_s) denote the stabilizer in $G^{der}(\mathbb{Q}_p)$ of its action on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p}))$ (respectively $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K, O_K))$ and $\pi_0(S_f)$).

By Chen's theorem 3.2.16 (phrased in terms of lemma 3.3.6) G_s is an open subgroup of $G^{der}(\mathbb{Q}_p)$ and we have inclusions $G^{der}_x, G^{der}_s \subseteq G^{der}_{x_K}$. By lemma 3.4.9, $G^{der}_x \cdot \mathcal{K}^{der} = G^{der}(\mathbb{Q}_p)$ which implies that $G^{der}_{x_K} \cdot \mathcal{K}^{der} = G^{der}(\mathbb{Q}_p)$ as well. In particular, the projection map $\mathcal{K}^{der} \to G^{der}(\mathbb{Q}_p)/G^{der}_{x_K}$ is surjective.

Since $G^{der}(\mathbb{Q}_p)/G^{der}_{x_K}$ has the discrete topology and $\mathcal{K}^{der}$ is compact, we get that $G^{der}_{x_K}$ is closed and of finite index within $G^{der}(\mathbb{Q}_p)$. Moreover, since G^{der} is quasi-split (even unramified) all of the simple factors of G^{der} are isotropic. By Margulis theorem [39] II 5.1 we can conclude that $G^{der}_{x_K} = G^{der}(\mathbb{Q}_p)$. Since the argument doesn't depend on the choice of x the action of $G^{der}(\mathbb{Q}_p)$ on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K, O_K))$ is trivial.

Now, $\mathrm{Spd}(C_p, O_{C_p}) = \varprojlim \mathrm{Spd}(K, O_K)$ and we may use [51] 11.22 to compute the action map

$$G^{der}(\mathbb{Q}_p) \times |\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p})| \to |\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p})|$$

as the limit of the action maps

$$\varprojlim_{K \subseteq C_p} [G^{der}(\mathbb{Q}_p) \times |\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K, O_K)| \to |\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K, O_K)|]$$

Since in the transition maps $|\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K_1, O_{K_1})| \to |\mathrm{Sht}_{G,b,[\mu],\infty} \mathrm{Spd}(K_2, O_{K_2})|$ every connected component on the source surjects onto a connected component on the target we have that $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p})) = \varprojlim \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(K, O_K))$. In particular, $G^{der}(\mathbb{Q}_p)$ acts trivially on the set of connected components. This defines a transitive action of $G^{ab}(\mathbb{Q}_p)$ on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C_p, O_{C_p}))$. The map $\pi_0(det)$ is surjective and equivariant for this action. Since $G^{ab}(\mathbb{Q}_p)$ acts freely on $\pi_0(\mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty} \times \mathrm{Spd}(C_p, O_{C_p}))$, $\pi_0(det)$ must be a bijection.

□

Corollary 3.4.11. *For G, b and μ as in theorem 3.4.10 and any compact subgroup $\mathcal{K} \subseteq G(\mathbb{Q}_p)$ the map*

$$\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} \to \mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],det(\mathcal{K})}$$

has non-empty connected geometric fibers.

Proof. One can deduce the claim for arbitrary compact $\mathcal{K}$ from the identity

$$\mathrm{Sht}_{G,b,[\mu],\mathcal{K}} = \mathrm{Sht}_{G,b,[\mu],\infty}/\underline{\mathcal{K}}.$$

Indeed, the formation of π_0 commutes with colimits, so that

$$\pi_0(\mathrm{Sht}_{G,b,[\mu],\mathcal{K}}) = \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty})/\mathcal{K}$$

which is $\pi_0(\mathrm{Sht}_{G^{ab},b^{ab},[\mu^{ab}],\infty})/det(\mathcal{K})$. □

Using functoriality and equivariance for the three actions we can describe the actions by the three groups on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times C_p)$ in the spirit of theorem 3.3.5.

Theorem 3.4.12. *(Compare with [7] 4.1) Let G, b and μ as in theorem 3.4.10. Let $E \subseteq C_p$ be the field of definition of $[\mu]$, let $Art_{\breve{E}/E} : W_{\breve{E}/E} \to E^\times$ be as in theorem 3.3.5, let $Nm_{\mu^{ab}} : \mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m) \to G^{ab}$ be the norm map associated to μ^{ab} then:*

1. *The $G(\mathbb{Q}_p)$ right action on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times C_p)$ makes it a trivial right $G^{ab}(\mathbb{Q}_p)$-torsor.*
2. *If $s \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times C_p)$ and $j \in J_b(\mathbb{Q}_p)$ then*

   $$s \cdot_{J_b(\mathbb{Q}_p)} j = s \cdot_{G^{ab}(\mathbb{Q}_p)} det(j^{-1}))$$

   where $det = j_{b^{ab}} \circ det_b$ with $det_b : J_b(\mathbb{Q}_p) \to J_{b^{ab}}(\mathbb{Q}_p)$ the map obtained from functoriality of the formation of J_b, respectively $J_{b^{ab}}$, and where the map $j_{b^{ab}}$ is the isomorphism $j_{b^{ab}} : J_{b^{ab}}(\mathbb{Q}_p) \cong G^{ab}(\mathbb{Q}_p)$ obtained from regarding $J_{b^{ab}}(\mathbb{Q}_p)$ as a subgroup of $G^{ab}(K_0)$.
3. *If $s \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times C_p)$ and $\gamma \in W_{\breve{E}/E}$ then*

   $$s \cdot_{W_{\breve{E}/E}} \gamma = s \cdot_{G^{ab}(\mathbb{Q}_p)} [Nm_{\mu^{ab}} \circ Art_{\breve{E}/E}(\gamma)].$$

3.4.3 z-extensions

In this subsection we extend theorem 3.4.10 to the case in which G is not necessarily simply connected but we still assume that G is unramified and (b, μ) is HN-irreducible. In what follows we will denote by G^{sc} the central simply connected cover of G^{der} and we denote by $G^\circ = G(\mathbb{Q}_p)/Im(G^{sc}(\mathbb{Q}_p))$. Notice that when G^{der} is simply connected $G^\circ = G^{ab}(\mathbb{Q}_p)$. In general, G° surjects onto $G^{ab}(\mathbb{Q}_p)$ and the kernel is a finite group.

Recall the following definition used extensively by Kottwitz:

Definition 3.4.13. *A map of connected reductive groups $f : G' \to G$ is a z-extension if:*

- *f is surjective.*
- *$Z = ker(f)$ is central in G'.*
- *Z is isomorphic to a product of tori of the form $\mathrm{Res}_{F_i/\mathbb{Q}_p}\mathbb{G}_m$ for some finite extensions $F_i \subseteq \overline{\mathbb{Q}}_p$.*

- *G' has simply connected derived subgroup.*

By [32] lemma 1.1 whenever G is an unramified group over $\mathbb{Q}_p$ that splits over $\mathbb{Q}_{p^s}$, there exists a z-extension $G' \to G$ with Z isomorphic to a product of tori of the form $\mathrm{Res}_{\mathbb{Q}_{p^s}/\mathbb{Q}_p}\mathbb{G}_m$. In particular, it is unramified as well.

In [35] Kottwitz proves that for any reductive group G and cocharacter μ the natural morphism $B(G) \to B(G^{ad})$ induces a bijection $B(G,\mu) \cong B(G^{ad},\mu^{ad})$. From here we can easily deduce the following:

Lemma 3.4.14. *Let $A \subseteq T \subseteq B \subseteq G$ as in the notation section. Assume that $\mathbb{Q}_{p^s}$ is a splitting field for G. Let $\mu \in X_*^+(T)$, $[b] \in B(G,\mu)$, and $f : G' \to G$ a z-extension with $Z = ker(f)$ isomorphic to a finite product of copies of $\mathrm{Res}_{\mathbb{Q}_{p^s}/\mathbb{Q}_p}\mathbb{G}_m$. Let $T' = f^{-1}(T)$ denote the maximal torus of G' projecting onto T. Then:*

1. *For any choice of $\mu' \in X_*(T')^+$ lifting μ there is a unique lift $[b'] \in B(G')$ lifting $[b]$ with $[b'] \in B(G',\mu')$.*

2. *For b' and μ' as in the previous claim (b,μ) is HN-irreducible if and only if (b',μ') is HN-irreducible.*

3. *If E is the field of definition of μ with $\mathbb{Q}_p \subseteq E \subseteq \mathbb{Q}_{p^s}$ then there is a lift $\mu' \in X_*(T')^+$ with field of definition E.*

Proof. The first claim follows directly from the identifications

$$B(G,\mu) = B(G^{ad},\mu^{ad}) = B(G',\mu').$$

The second claim follows from the first claim, from the fact that $Z := ker(f)$ is central and from the fact that HN-irreducibility can be checked on the adjoint quotient once it is known that $b' \in B(G,\mu')$ holds.

For the third claim consider the exact sequence of $\Gamma_{\mathbb{Q}_p}$-modules:

$$e \to X_*(Z) \to X_*(T') \to X_*(T) \to e$$

Since G and G' split over $\mathbb{Q}_{p^s}$ the subgroup $\Gamma_{\mathbb{Q}_{p^s}} \subseteq \Gamma_E \subseteq \Gamma_{\mathbb{Q}_p}$ acts trivially on all of these groups. We treat this as an exact sequence of $Gal(\mathbb{Q}_{p^s}/E)$-modules. Since $Z = \prod_{i=1}^n \mathrm{Res}_{\mathbb{Q}_{p^s}/\mathbb{Q}_p}(\mathbb{G}_m)$ for some n, we can conclude that $X_*(Z)$ is an induced $\mathbb{Z}[Gal(\mathbb{Q}_{p^s}/E)]$-module and by Shapiro's lemma $H^1(Gal(\mathbb{Q}_{p^s}/E), X_*(Z)) = 0$. This implies that

$$X_*(T')^{\Gamma_E} = X_*(T')^{Gal(\mathbb{Q}_{p^s}/E)} \to X_*(T)^{Gal(\mathbb{Q}_{p^s}/E)} = X_*(T)^{\Gamma_E}$$

is surjective as we wanted to prove.

□

Proposition 3.4.15. *Suppose that G' is an unramified group, (b',μ') a pair with $[b'] \in B(G',\mu')$, suppose that $Z \subseteq G'$ is a central torus, and let $G = G'/Z$ with projection map $f : G' \to G$. Let $b = f(b')$ and $\mu' = f \circ \mu$ the following hold:*

1. $Gr^{\leq[\mu']}(\mathcal{E}_{b'}) \to Gr^{\leq[\mu]}(\mathcal{E}_b)$ *is an isomorphism.*

2. $Gr^{adm,\leq[\mu']}(\mathcal{E}_{b'}) \to Gr^{adm,\leq[\mu]}(\mathcal{E}_b)$ *is an isomorphism.*

3. *If* $\mathbb{L}_{G'}$ *(respectively* $\mathbb{L}_G$*) denotes the pro-étale* $\underline{G'(\mathbb{Q}_p)}$*-torsor (respectively* $\underline{G(\mathbb{Q}_p)}$*-torsor) then* $\mathbb{L}_G = f_*\mathbb{L}_{G'}$*.*

Proof. Both $Gr^{\leq[\mu']}(\mathcal{E}_{b'})$ and $Gr^{\leq[\mu]}(\mathcal{E}_b)$ are spatial diamonds that are proper over $\mathrm{Spd}(K_0, O_{K_0})$, any morphism between them is qcqs and by [51] 12.5 it is enough to prove the map is a bijection at the level of geometric points. In this case after fixing an isomorphism $B_{dR}(C) \cong C((t))$ we may reason as in the classical case. That is,

$$Gr(\mathcal{E}_{b'})(C, C^+) \cong G'(C((t)))/G'(C[[t]]),$$

also

$$Gr(\mathcal{E}_b)(C, C^+) \cong G(C((t)))/G(C[[t]])$$

and the map

$$G'(C((t)))/G'(C[[t]]) \to G(C((t)))/G(C[[t]])$$

is a $Z(C((t)))/Z(C[[t]])$-torsor. On the other hand $Z(C((t)))/Z(C[[t]]) \cong X_*(Z)$ and we have an exact sequence:

$$e \to X_*(Z) \to X_*(T') \to X_*(T) \to e,$$

and the lifts of μ also form a $X_*(Z)$-torsor. Given a point $x \in Gr(\mathcal{E}_b)(C, C^+)$ of type $\mu \in X_*^+(T)$ and a lift $\mu'' \in X_*^+(T')$ there is a unique $y \in Gr(\mathcal{E}_b)(C, C^+)$ of type μ'' this finishes the proof of the first claim.

Let us prove the second claim, by the previous claim $Gr^{adm,\leq[\mu']}(\mathcal{E}_{b'})$ and $Gr^{adm,\leq[\mu]}(\mathcal{E}_b)$ are two open sub-diamonds of $Gr^{\leq[\mu]}(\mathcal{E}_b)$. By [51] 11.15 it is enough to understand the underlying topological space of this open subsheaves. We prove that $Gr^{adm,\leq[\mu']}(\mathcal{E}_{b'})(C, C^+) \to Gr^{adm,\leq[\mu]}(\mathcal{E}_b)(C, C^+)$ is a bijection.

If we represent an element $x \in Gr^{\leq[\mu']}(\mathcal{E}_{b'})(C, C^+)$ by a modification $(\alpha_x : \mathcal{E}_x \dashrightarrow \mathcal{E}_{b'})$, then $f(x)$ is represented by $(f_*\alpha_x : f_*\mathcal{E}_x \dashrightarrow \mathcal{E}_b)$. By definition $x \in Gr^{adm,\leq[\mu']}(\mathcal{E}_{b'})(C, C^+)$ when $\mathcal{E}_x$ is a trivial G'-torsor this implies $f_*\mathcal{E}_x$ is trivial so that $f(x) \in Gr^{adm,\leq[\mu]}(\mathcal{E}_b)(C, C^+)$. Assume instead $f(x) \in Gr^{adm,\leq[\mu]}(\mathcal{E}_b)(C, C^+)$, and let $[b'_x] \in B(G')$ be the unique element with $\mathcal{E}_{b'_x} \cong \mathcal{E}_x$. We need to prove $[b'_x] = [e]$. We begin by proving that $\kappa([b'_x]) = \kappa([b']) - [\mu']$. Indeed using ([14] 2.15) we can deduce that $\kappa(\mathcal{E}_x)$ is independent of $x \in Gr^{\leq[\mu]}(\mathcal{E}_{b'})(C, C^+)$ since $Gr^{\leq[\mu]}(\mathcal{E}_{b'})$ is connected. It is then enough to prove $\kappa([b'_x]) = \kappa([b']) - [\mu']$ when $x \in Gr^{\leq[\mu]}(\mathcal{E}_{b'})(C, C^+)$ is the point associated to ξ^μ. This is precisely the content of ([27] 6.4.1).

By the assumption $b' \in B(G', \mu')$ we have $\kappa([b'_x]) = [e] \in \pi_1(G')_{\Gamma_{\mathbb{Q}_p}}$, so that to prove $[b'_x] = [e]$ it is enough to prove that $[b'_x]$ is basic. But $f([b'_x]) = [e]$ so $\nu_{b'_x}$ must factor through $X_*(Z) \otimes \mathbb{Q}$, and since Z is central $[b'_x]$ is basic.

For the last claim, recall that for any $(V, \rho) \in Rep_{G'}(\mathbb{Q}_p)$ and $x \in Gr^{adm,\leq[\mu]}(\mathcal{E}_{b'})(R, R^+)$, $\rho_*\mathbb{L}_{G'}(x)$ evaluates to $\underline{H}^0(\mathcal{X}_{FF,R}, \rho_*\mathcal{E}_x)$. When $\rho = \tau \circ f$ we get $\underline{H}^0(\mathcal{X}_{FF,R}, \tau_*\mathcal{E}_{f(x)})$ which is the evaluation of $\mathbb{L}_G$ at $(V,\tau) \in Rep_G(\mathbb{Q}_p)$. $\square$

Proposition 3.4.16. *If (b,μ) is HN-irreducible then the following hold:*

1. *$Gr^{adm,\leq[\mu]}(\mathcal{E}_b) \times \mathrm{Spd}(C,O_C)$ is connected*

2. *The right action of $G(\mathbb{Q}_p)$ on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C,O_C))$ makes this set into a G°-torsor.*

Proof. Using lemma 3.4.14 we may find a z-extension $f : G' \to G$ and lift (b,μ) to a pair (b',μ') over G' which is also HN-irreducible. The first claim now follows from proposition 3.4.15 and by proposition 3.4.8 applied to G', since by definition of z-extension $(G')^{der}$ is simply connected.

Let $Z = Ker(f)$, since this is an induced torus Hilbert's 90 theorem together with Shapiro's lemma proves the surjectivity of the map $f : G'(\mathbb{Q}_p) \to G(\mathbb{Q}_p)$. Using this together with proposition 3.4.15 we see that

$$f : \mathrm{Sht}_{G',b',[\mu'],\infty} \times \mathrm{Spd}(C,O_C) \to \mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C,O_C)$$

is $\underline{Z(\mathbb{Q}_p)}$-torsor. In particular, the map of sets of connected components is also surjective. Since $Gr^{adm,\leq[\mu]}(\mathcal{E}_b)$ is connected the action of $G(\mathbb{Q}_p)$ on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C,O_C))$ is transitive. Let $x \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C,O_C))$ and denote by G_x the stabilizer of x in $G(\mathbb{Q}_p)$. Let $y \in \pi_0(\mathrm{Sht}_{G',b',[\mu'],\infty} \times \mathrm{Spd}(C,O_C))$ a lift of x, we wish to prove that $Im(G^{sc}(\mathbb{Q}_p)) = G_x$.

By theorem 3.4.10 the stabilizer of y in $G'(\mathbb{Q}_p)$ is $(G')^{der}(\mathbb{Q}_p)$. By equivariance of f with respect the actions of $G'(\mathbb{Q}_p)$ and $G(\mathbb{Q}_p)$, we have that $Im((G')^{der}(\mathbb{Q}_p)) \subseteq G_x$. Since G' is a z-extension $Im((G')^{der}(\mathbb{Q}_p)) = Im(G^{sc}(\mathbb{Q}_p))$. On the other hand, any $g \in G_x$ has a lift $g' \in G'(\mathbb{Q}_p)$ and we may write $f(y \cdot g') = x \cdot g = x$. Since $f(y \cdot g') = f(y)$, there is an element $z \in Z(\mathbb{Q}_p)$ with $y \cdot g' \cdot z = y$. In other words, $z \cdot g' \in (G')^{der}(\mathbb{Q}_p)$ which implies that $g \in Im(G^{sc}(\mathbb{Q}_p))$ finishing the proof. $\square$

As we have done in previous subsections we can describe the action of $J_b(\mathbb{Q}_p)$ and $W_{\check{E}/E}$ on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C,O_C))$ in terms of the action of G°. We first describe the action of $J_b(\mathbb{Q}_p)$. To do this we need to construct a map $det^\circ : J_b(\mathbb{Q}_p) \to G^\circ$ that generalizes the determinant map $det : J_b(\mathbb{Q}_p) \to G^{ab}(\mathbb{Q}_p)$ that appears in theorem 3.4.12. A peculiar aspect of the situation is that G° does not necessarily have algebraic structure (its not the $\mathbb{Q}_p$-points of an algebraic group). Consequently det° is does not come directly from a map of algebraic groups. The map is constructed as follows: Given G and $b \in G(K_0)$ we may choose an unramified z-extension $f : G' \to G$ and a lift $b' \in G'(K_0)$ with $f(b') = b$. Let $Z = Ker(f)$. We get a sequence of maps of reductive groups

$$e \to Z \to J_{b'} \to J_b \to e.$$

Since Z is an induced torus, by Hilbert's theorem 90 and Shapiro's lemma $H^1(\mathbb{Q}_p, Z) = \{0\}$ so that we obtain a surjection $J_{b'}(\mathbb{Q}_p) \to J_b(\mathbb{Q}_p)$. We can construct the following commutative diagram of topological groups:

$$\begin{tikzcd}
Z(\mathbb{Q}_p) \arrow[r] \arrow[d] & G'(\mathbb{Q}_p) \arrow[r, "f"] \arrow[d] & G(\mathbb{Q}_p) \arrow[d] \\
J_{b'}(\mathbb{Q}_p) \arrow[r, "det"] \arrow[dr] \arrow[dd, "f"] & (G')^{ab}(\mathbb{Q}_p) \arrow[r, "f^{ab}"] & G^\circ \\
 & J_{b'^{ab}}(\mathbb{Q}_p) \arrow[u, "\cong_j"] & \\
J_b(\mathbb{Q}_p) \arrow[uurr, bend right, "det^\circ"'] & &
\end{tikzcd}$$

Now, det° is defined as the unique morphism that could make this diagram commutative. More explicitly, if $j \in J_b(\mathbb{Q}_p)$ we pick a lift $j' \in J_{b'}(\mathbb{Q}_p)$, and we define $det^\circ(j) := f^{ab}(det(j'))$. This doesn't depend on the choice of j'. Indeed, two lifts of j differ by an element of $Z(\mathbb{Q}_p)$ but the induced map $Z(\mathbb{Q}_p) \to G^\circ$ is the 0 map, since it factors through the map to G. Similarly the construction of det° does not depend of the choice of $b' \in G'(\mathcal{K}_0)$ lifting b since the possible choices differ by an element of $Z(K_0)$. Finally, we justify that the construction of det° doesn't depend on the choice of z-extension $G' \to G$ taken. This will follow from the fact that the category of z-extensions of G is cofiltered. Given two z-extensions $G_1, G_2 \to G$ we may find a third z-extension making the following diagram commutative:

$$\begin{tikzcd}
 & G_2 \arrow[dr] & \\
G_3 \arrow[ur, "f_2"] \arrow[dr, "f_1"] & & G \\
 & G_1 \arrow[ur] &
\end{tikzcd}$$

Choosing a lift of $b_3 \in G_3(K_0)$ and defining $b_i = f_i(b_3)$ we obtain the following diagram:

$$\begin{tikzcd}
J_{b_3}(\mathbb{Q}_p) \arrow[r, "f_i"] \arrow[d, "det"] & J_{b_i}(\mathbb{Q}_p) \arrow[r] \arrow[d, "det"] & J_b(\mathbb{Q}_p) \arrow[d, bend right, "det_i^\circ"'] \arrow[d, bend left, "det_3^\circ"] \\
G_3^{ab}(\mathbb{Q}_p) \arrow[r] & G_i^{ab}(\mathbb{Q}_p) \arrow[r] & G^\circ
\end{tikzcd}$$

It is easy to verify $det_i^\circ = det_3^\circ$.

Remark 3.4.17. *Another way one can define* $\det^\circ$ *is as follows. Since* G *is quasi-split we may define groups* $A \subseteq T \subseteq B \subseteq G$ *as in the notation section §2.2. The dominant Newton point* ν_b^{dom} *is a* $\mathbb{Q}_p$*-rationally defined map* $\mathbb{D} \to A$ *and we may define* M_b *as the centralizer of* ν_b *in* G*. One may then reconstruct* J_b *as a twisted inner form of* M_b*. Using z-extensions one may construct an isomorphism from* $J_b(\mathbb{Q}_p)/[J_b(\mathbb{Q}_p), J_b(\mathbb{Q}_p)]$ *and* $M_b(\mathbb{Q}_p)/[M_b(\mathbb{Q}_p), M_b(\mathbb{Q}_p)]$ *(the maximal abelian quotients when regarded as an abstract* *groups). The inclusion $M_b(\mathbb{Q}_p) \subseteq G(\mathbb{Q}_p)$ induces a map $M_b(\mathbb{Q}_p) \to G^\circ$ which overall gives a map $J_b(\mathbb{Q}_p) \to G^\circ$. Again, one must justify that this morphism didn't depend of the choices made.*

By functoriality, equivariance and theorem 3.4.12 we can do the following computation. Pick G', b' and μ' as in the proof of proposition 3.4.16. We obtain a map

$$f : \mathrm{Sht}_{G',b',[\mu'],\infty} \times \mathrm{Spd}(C, O_C) \to \mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C, O_C),$$

let $x \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C, O_C))$ and let $y \in \pi_0(\mathrm{Sht}_{G',b',[\mu'],\infty} \times \mathrm{Spd}(C, O_C))$ be a lift of x. Let $j \in J_b(\mathbb{Q}_p)$, and let $j' \in J_{b'}(\mathbb{Q}_p)$ be an element lifting j. We have:

$$\begin{aligned} x \cdot_{J_b(\mathbb{Q}_p)} j &= f(y \cdot_{J_{b'}(\mathbb{Q}_p)} j') \\ &= f(y \cdot_{G'(\mathbb{Q}_p)} j_{b'}(det_{b'}(j^{-1}))) \\ &= x \cdot_{G^\circ} det^\circ(j^{-1}) \end{aligned}$$

We now describe the action of $W_{\breve{E}/E}$, we will also need to introduce a variant of the norm map discussed for tori. Given a connected reductive group G and a conjugacy class of cocharacters $[\mu]$ with reflex field E we define a norm map $Nm^\circ_{[\mu]} : E^\times \to G^\circ$ as follows. Since is G is quasi-split we may fix $\mathbb{Q}_p$-rationally defined Borel a maximal torus $T \subseteq B \subseteq G$ and the unique dominant cocharacter $\mu \in X_*^+(T)$ representing $[\mu]$ and defined over E. We get a norm map $Nm_\mu : E^\times \to T(\mathbb{Q}_p)$ and we may define $Nm^\circ_{[\mu]}$ as the composition:

$$Nm^\circ_{[\mu]} : E^\times \xrightarrow{Nm_\mu} T(\mathbb{Q}_p) \to G(\mathbb{Q}_p) \to G^\circ.$$

We claim that this map is independent of the choice of B and T. Indeed, recall that the action of $G(\mathbb{Q}_p)$ on the set of pairs (B, T) with B a rationally defined Borel and T a rationally defined maximal torus contained in B is transitive. If $(B_2, T_2) = g \cdot (B_1, T_1) \cdot g^{-1}$ for some element $g \in G(\mathbb{Q}_p)$ then $Nm^\circ_{g\cdot\mu\cdot g^{-1}} = g \cdot Nm_\mu g^{-1}$, and since G° is abelian we get $Nm^\circ_{[g\cdot\mu\cdot g^{-1}]} = Nm^\circ_{[\mu]}$.

Proposition 3.4.18. *With notation as in proposition 3.4.16 the action of $W_{\breve{E}/E}$ on*

$$\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C, O_C))$$

is given by the map $Nm^\circ_{[\mu]} \circ Art_{\breve{E}/E} : W_{\breve{E}/E} \to G^\circ$. More precisely, if

$$x \in \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C, O_C))$$

and $\gamma \in W_{\breve{E}/E}$ then:

$$x \cdot_{W_{\breve{E}/E}} \gamma = x \cdot_{G^\circ} Nm^\circ_{[\mu]}(Art_{\breve{E}/E}(\gamma)).$$

Proof. We let $f : G' \to G$ be a z-extension and we let (b', μ') be a pair over G' lifting (b, μ), and let $Z = ker(f)$. By 3.4.14 we can always choose G' and μ' so that μ' has the same field of definition as μ. We get a morphism

$$\mathrm{Sht}_{G',b',[\mu'],\infty} \to \mathrm{Sht}_{(G')^{ab},b'^{ab},[\mu'],\infty}.$$

Let $A \subseteq T \subseteq B \subseteq G$ as above and let $T' = f^{-1}(T)$. Recall that for tori the set $B(T',\mu')$ has a unique element, we fix a representative $b_{\mu'}$. This allows us to construct a map

$$\mathrm{Sht}_{T',b_{\mu'},[\mu'],\infty} \to \mathrm{Sht}_{(G')^{ab},b'^{ab},[\mu'],\infty}$$

and by functoriality we also get map

$$\mathrm{Sht}_{T',b_{\mu'},[\mu'],\infty} \to \mathrm{Sht}_{T,b_{\mu},[\mu],\infty}$$

We can collect all of these maps in the following commutative diagram of spaces.

$$\begin{tikzcd}
\mathrm{Sht}_{G',b',[\mu'],\infty} \arrow[r] \arrow[d] & \mathrm{Sht}_{(G')^{ab},b'^{ab},[\mu'],\infty} & \mathrm{Sht}_{T',b_{\mu'},[\mu'],\infty} \arrow[l] \arrow[d] \\
\mathrm{Sht}_{G,b,[\mu],\infty} & & \mathrm{Sht}_{T,b_{\mu},[\mu],\infty}
\end{tikzcd}$$

Since G' is simply connected we get an equivariant bijection of geometric connected components

$$\pi_0(\mathrm{Sht}_{G',b',[\mu'],\infty} \times \mathrm{Spd}(C,O_C)) \to \pi_0(\mathrm{Sht}_{(G')^{ab},b'^{ab},[\mu'^{ab}],\infty} \times \mathrm{Spd}(C,O_C)).$$

After forming geometric connected components and choosing a base point

$$x \in \pi_0(\mathrm{Sht}_{T',b_{\mu'},[\mu'],\infty} \times \mathrm{Spd}(C,O_C))$$

the above diagram looks like this:

$$\begin{tikzcd}
x \cdot G'^{ab}(\mathbb{Q}_p) \arrow[r, "\cong"] \arrow[d] & x \cdot G'^{ab}(\mathbb{Q}_p) & x \cdot T'(\mathbb{Q}_p) \arrow[l] \arrow[d] \\
x \cdot G^{\circ} & & x \cdot T(\mathbb{Q}_p)
\end{tikzcd}$$

All of the maps are equivariant with respect to the groups involved. Since the map $T'(\mathbb{Q}_p) \to G^{\circ}$ factors through the map $T'(\mathbb{Q}_p) \to T(\mathbb{Q}_p)$, we get a canonical surjective and $W_{\breve{E}/E}$-equivariant map

$$\pi_0(\mathrm{Sht}_{T,b_{\mu},[\mu],\infty} \times \mathrm{Spd}(C,O_C)) \to \pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C,O_C)).$$

By theorem 3.3.5, the action on $\pi_0(\mathrm{Sht}_{T,b_{\mu},[\mu],\infty})$ is through $Nm_{\mu} \circ Art_{\breve{E}/E}$. Equivariance and the definition of $Nm^{\circ}_{[\mu]}$ imply that the action of $W_{\breve{E}/E}$ on $\pi_0(\mathrm{Sht}_{G,b,[\mu],\infty} \times \mathrm{Spd}(C,O_C))$ is through $Nm^{\circ}_{[\mu]} \circ Art_{\breve{E}/E}$. $\square$

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