MEROMORPHIC VECTOR BUNDLES ON THE FARGUES–FONTAINE CURVE

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ABSTRACT. We introduce and study the stack of *meromorphic* G-bundles on the Fargues–Fontaine curve. This object defines a correspondence between the Kottwitz stack $\mathfrak{B}(G)$ and Bun_G. We expect it to play a crucial role in defining and studying an analytification functor that compares the schematic and analytic versions of the geometric local Langlands categories. Our first main result is the identification of the generic Newton strata of $\mathrm{Bun}_G^{\mathrm{mer}}$ with the Fargues–Scholze charts $\mathcal{M}$. Our second main result is a generalization of Fargues' theorem in families. We call this the *meromorphic comparison theorem*. We expect it to play a key role in proving that the analytification functor is fully-faithful. Along the way, we give new proofs to what we call the *topological and schematic comparison theorems*. These say that the topologies of Bun_G and $\mathfrak{B}(G)$ are reversed and that the two stacks take the same values when evaluated on schemes.

CONTENTS

1. INTRODUCTION

Let p be a prime number, let E be a non-Archimedean local field with residue field of characteristic p. Let ℓ be a prime with $\ell \neq p$, and $\Lambda = \overline{\mathbb{Q}}_\ell$. Let G be a connected reductive group over E. Let W_E be the Weil group and let ${}^L G = \hat{G} \rtimes W_E$ be the L-group. For this introduction, we will further assume that G is quasi-split, but we drop this assumption in the body of the text.

1.1. **Motivation and context.** Let Π_G be the set of isomorphism classes of smooth irreducible representations of the locally profinite group $G(E)$ with values in Λ and let Φ_G be the set of $\hat{G}$-conjugacy classes of L-parameters. The basic form of the local Langlands correspondence gives a map

$$\mathrm{LLC}_G : \Pi_G \to \Phi_G$$

satisfying various desiderata [Kal16, Conjecture A], [SZ18]. For GL_n, the map $\mathrm{LLC}_{\mathrm{GL}_n}$ is bijective [HT01, Hen00], but this does not hold more generally. Nevertheless, LLC_G has finite fibers that are called L-packets and understanding them is the subject of the refined local Langlands correspondence.

For quasi-split groups, one can fix a Whittaker datum $\mathfrak{w}$ to put the elements of an L-packet in canonical bijection with the set of isomorphisms classes of irreducible representations of a certain finite group, which is constructed in terms of the L-parameter [Kal16, Conjecture B]. When G is not quasi-split, Whittaker data do not exist. Vogan realized that to work with general G, it is advantageous to consider its quasi-split inner form G^* and parametrize simultaneously the representations of all the pure inner twists of G^* [ABV92, Vog93].

Motivated by the study of special fibers of Shimura varieties, Kottwitz introduced the set $B(G)$ of isocrystals with G-structure [Kot85, Kot97]. The set of basic elements $B(G)_{\mathrm{bas}} \subseteq B(G)$ gives rise to the so-called extended pure inner forms G_b of G. Kottwitz formulated a refined version of the local Langlands correspondence for non-Archimedean local fields using the inner forms that arise from $B(G)_{\mathrm{bas}}$ [Kal16, Conjecture F], [SZ18].

The set $B(G)$ can be realized as the underlying topological space of two geometric objects. One object is of analytic nature, Bun_G (the stack of G-bundles on the Fargues–Fontaine curve), and a second object is of schematic nature, $\mathfrak{B}(G)$ (the Kottwitz stack parametrizing isocrystals with G-structure). For every element $b \in B(G)$ one can define locally closed strata $i_b : \mathfrak{B}(G)_b \to \mathfrak{B}(G)$ and $j_b : \mathrm{Bun}_G^b \to \mathrm{Bun}_G$. Both $\mathfrak{B}(G)_b$ and Bun_G^b are classifying stacks for a group, and sheaves on these classifying stacks can be described in terms of smooth representations of $G_b(E)$, where G_b is an inner form of a Levi subgroup of G. This leads to the hope that the refined local Langlands correspondence of Kottwitz has a categorical refinement that one can access by studying the geometry of the stacks Bun_G and/or $\mathfrak{B}(G)$.

Recent breakthroughs in p-adic and perfect geometry [SW20, FS24, Zhu17, XZ17, BS17, Zhu20] together with the introduction and study of the stack of L-parameters [DHKM20, Zhu20, FS24] have led experts to formulate precise conjectures that capture this hope. These efforts promote the refined local Langlands correspondence mentioned above to a categorical statement [FS24, Zhu20, Hel23, BZCHN22] in a precise way.

There is widespread agreement on what to consider on the Galois side, namely a version of the derived category of coherent sheaves $\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(\mathcal{X}_{\hat{G},\Lambda})$ of the stack $\mathcal{X}_{\hat{G},\Lambda}$ parametrizing L-parameters over Λ (see [FS24, Conjecture I.10.2], [AG15]). On the automorphic side there are at least two reasonable constructions of the local Langlands category. The essential difference between them arises from the fact that $B(G)$ has two geometric incarnations.

Let G be quasi-split and let $W_{\mathfrak{w}}$ be the Whittaker representation associated to $\mathfrak{w}$. On the analytic side, Fargues–Scholze construct the category of lisse sheaves $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ [FS24, § VII.7] and prove it is compactly generated. In what follows, we use the superscript $(-)^{\omega}$ to denote the full subcategory of compact objects. Moreover, they endow this category with the so-called spectral action by the category of perfect complexes $\mathrm{Perf}(\mathcal{X}_{\hat{G},\Lambda})$. They conjecture that there is a unique $\mathrm{Perf}(\mathcal{X}_{\hat{G},\Lambda})$-linear equivalence of ∞-categories

$$\mathbb{L}^{\mathrm{an}}_G : \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{\omega} \xrightarrow{\simeq} \mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(\mathcal{X}_{\hat{G},\Lambda})$$

which, by taking ind-completions, induces an equivalence $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \xrightarrow{\simeq} \mathrm{Ind}(\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(\mathcal{X}_{\hat{G},\Lambda}))$ that sends the analytic Whittaker sheaf $\mathcal{W}^{\mathrm{an}}_{\mathfrak{w}} = j_{1,!}W_{\mathfrak{w}}$ to the structure sheaf $\mathcal{O}_{\mathcal{X}_{\hat{G},\Lambda}}$.

On the schematic side, Xiao–Zhu consider the moduli stack of local shtukas $\mathrm{Sht}^{\mathrm{loc}}_k$ in the context of characteristic p perfect geometry. They attach their own candidates for the local Langlands category, namely they construct a triangulated category of cohomological correspondences $\mathrm{P}^{\mathrm{Corr}}(\mathrm{Sht}^{\mathrm{loc}}_k)$, cf. [XZ17, § 5.4] and [Zhu20]. This approach is pushed further in the forthcoming work of Hemo–Zhu [HZ], where they construct an ∞-category $\mathrm{Shv}(\mathfrak{B}(G), \Lambda)$ whose homotopy category agrees with $\mathrm{P}^{\mathrm{Corr}}(\mathrm{Sht}^{\mathrm{loc}}_k)$. Zhu conjectures that there is an equivalence

$$\mathbb{L}^{\mathrm{sch}}_G : \ \mathrm{Shv}(\mathfrak{B}(G), \Lambda) \xrightarrow{\simeq} \mathrm{Ind}(\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(\mathcal{X}_{\hat{G},\Lambda}))\,,$$

sending $\mathcal{O}_{\mathcal{X}_{\hat{G},\Lambda}}$ to the schematic Whittaker sheaf $\mathcal{W}^{\mathrm{sch}}_{\mathfrak{w}} = i_{1,*}W_{\mathfrak{w}}$, see [Zhu20, Conjecture 4.6.4]. Moreover, Hemo–Zhu have announced a proof of the unipotent part of the categorical local Langlands correspondence, cf. [Zhu20, Theorem 4.6.11]. Let us clarify. When Λ is of characteristic 0, the stack of

L-parameters has an open and closed substack $\mathcal{X}^{\mathrm{unip}}_{\hat{G},\Lambda} \subseteq \mathcal{X}_{\hat{G},\Lambda}$ defining a full subcategory

$$\mathrm{Ind}(\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(\mathcal{X}^{\mathrm{unip}}_{\hat{G},\Lambda})) \subseteq \mathrm{Ind}(\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(\mathcal{X}_{\hat{G},\Lambda}))\,.$$

One can also define a full subcategory $\mathrm{Shv}^{\mathrm{unip}}(\mathfrak{B}(G),\Lambda) \subseteq \mathrm{Shv}(\mathfrak{B}(G),\Lambda)$ defined by the property that for all $b \in B(G)$, the restriction to $\mathfrak{B}(G)_b$ is given by a complex of G_b-representations that are unipotent in the sense of Lusztig [Lus95]. Using Bezrukavnikov's equivalence [Bez16], Hemo and Zhu prove that there is an equivalence of ∞-categories

$$\mathbb{L}^{\mathrm{sch}}_G\colon\ \mathrm{Shv}^{\mathrm{unip}}(\mathfrak{B}(G),\Lambda) \xrightarrow{\simeq} \mathrm{Ind}(\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(\mathcal{X}^{\mathrm{unip}}_{\hat{G},\Lambda}))\,.$$

It is natural to expect that there exists an equivalence

$$\Psi\colon\ \mathrm{Shv}(\mathfrak{B}(G),\Lambda) \to \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)\,,$$

satisfying $\Psi(\mathcal{W}^{\mathrm{sch}}_{\mathfrak{w}}) = \mathcal{W}^{\mathrm{an}}_{\mathfrak{w}}$. Indeed, the two local Langlands categories are conjectured to be equivalent to $\mathrm{Ind}(\mathcal{D}^{b,\mathrm{qc}}_{\mathrm{coh}}(\mathcal{X}_{\hat{G},\Lambda}))$, and if the two conjectures are true one can simply define $\Psi = \mathbb{L}^{\mathrm{an},-1}_G \circ \mathbb{L}^{\mathrm{sch}}_G$.

A reasonable question the reader can ask is: why do we need two local Langlands categories? We believe that it is profitable to construct Ψ directly in order to better understand $\mathbb{L}^{\mathrm{sch}}_G$ and $\mathbb{L}^{\mathrm{an}}_G$. At a technical level, a direct construction of Ψ allows one to transfer Hemo–Zhu's results on unipotent categorical local Langlands correspondence to the Fargues–Scholze setup and conversely, endow $\mathrm{Shv}(\mathfrak{B}(G),\Lambda)$ with a spectral action. It would also allow us to formulate rigorously the eigensheaf property for the Deligne–Lusztig sheaves considered in [CI23, Conjecture 9.6]. More philosophically, the schematic perspective and the analytic perspective understand different phenomena. For example, the schematic perspective cannot witness the spectral action because "the paw" is fixed. On the other hand, $\mathrm{Shv}(\mathfrak{B}(G),\Lambda)$ is directly related to Bezrukavnikov's equivalence and its Frobenius-twisted categorical trace [Zhu18, §3] since, in contrast with Bun_G, both $\mathfrak{B}(G)$ and the Hecke stack are constructed in terms of Witt vector loop groups.

At the heart of the equivalence Ψ there should be a geometric explanation. Namely, that the stacks $\mathfrak{B}(G)$ and Bun_G are incarnations of the same geometric object. In this paper, we reveal these geometric relations which we formulate in terms of three comparison theorems (see §7).

One of the achievements of this article is the construction of a third incarnation $\mathrm{Bun}^{\mathrm{mer}}_G$ that mediates between $\mathfrak{B}(G)$ and Bun_G. Roughly speaking, $\mathrm{Bun}^{\mathrm{mer}}_G$ is given by the same moduli problem as Bun_G, but we require a meromorphicity condition on the action of Frobenius (see Definition 4.19, Definition 6.5). This object defines a correspondence

$$\begin{array}{ccc} \mathrm{Bun}^{\mathrm{mer}}_G & \xrightarrow{\sigma} & \mathrm{Bun}_G \\ \big\downarrow{\scriptstyle\gamma} & & \\ \mathfrak{B}(G)^{\diamondsuit} & & \end{array}$$

in the category of v-stacks on perfectoid spaces in characteristic p, where $(-)^{\diamondsuit}$ denotes a suitable analytification of $\mathfrak{B}(G)$, cf. Definition 2.5. We call the map γ the *generic polygon map* and σ the *special polygon map* inspired by [KL13, Definition 7.4.1]. Morally, Ψ should be given by $\sigma_! \circ \gamma^* \circ c^*$, where

$$c^*\colon\ \mathrm{Shv}(\mathfrak{B}(G),\Lambda) \to \mathcal{D}(\mathfrak{B}(G)^{\diamondsuit},\Lambda)$$

is an analytification functor [Sch17, §27], but we will not pursue the construction of this functor here.

1.2. **Main results.** For $b \in B(G)$ we let $\mathfrak{B}(G)_b \subseteq \mathfrak{B}(G)$ denote the locally closed substack determined by b. Then $\mathfrak{B}(G)^{\diamondsuit}_b \subseteq \mathfrak{B}(G)^{\diamondsuit}$ is also a locally closed substack and we have an identification

$$\mathfrak{B}(G)^{\diamondsuit}_b = [*/\underline{G_b(E)}]\,.$$

Recall the moduli stack $\mathcal{M}$ of Fargues–Scholze [FS24, Definition V.3.2] that is used to define the smooth charts of Bun_G. It comes endowed with a map

$$q\colon \mathcal{M} \to \coprod_{b\in B(G)} [*/\underline{G_b(E)}] \cong \coprod_{b\in B(G)} \mathfrak{B}(G)_b^{\diamond}.$$

The following Theorem 1.1 is a relative and Tannakian version of Kedlaya's work on the slope filtration [Ked05, Section 5.4], and our first main result.

Theorem 1.1 (Theorem 6.13)**.** *We have a commutative diagram with Cartesian square*

$$\begin{tikzcd} \mathcal{M} \arrow[r] \arrow[d, "q"] \arrow[rr, bend left, "\pi"] & \mathrm{Bun}_G^{\mathrm{mer}} \arrow[r, "\sigma"] \arrow[d, "\gamma"] & \mathrm{Bun}_G . \\ \coprod_{b\in B(G)} \mathfrak{B}(G)_b^{\diamond} \arrow[r] & \mathfrak{B}(G)^{\diamond} & \end{tikzcd}$$

In other words, the restriction of $\sigma\colon \mathrm{Bun}_G^{\mathrm{mer}} \to \mathrm{Bun}_G$ to $\gamma^{-1}([*/\underline{G_b(E)}])$ coincides with the Fargues–Scholze chart $\pi_b\colon \mathcal{M}_b \to \mathrm{Bun}_G$ [FS24, V.3].

Remark 1.2. The proof of Theorem 1.1 is done by discussing the vector bundle case in length and appealing to Tannakian formalism to prove the statement for general groups G. To apply Tannakian formalism one has to take subtle care of the exact structure. We do this by justifying that a sequence of meromorphic vector bundles is exact if and only if it is exact at every geometric point (see Proposition 4.21).

Remark 1.3. Z. Wu also proved a version of Theorem 6.13 independently (see Remark 6.14).

Theorem 6.13 should be closely related to the essential surjectivity of Ψ. Our second main result, which we now explain, should be related to fully-faithfulness. Recall the analytification functor $X \mapsto X^{\dagger}$, obtained from sheafifying the formula

$$(R, R^+) \mapsto X(\mathrm{Spec}\, R^{\circ}),$$

see Definition 2.5. For any small v-stack X we have fully-faithful maps

$$\mathcal{D}_{\acute{e}t}(X, \mathbb{F}_\ell) \xrightarrow{c_X^*} \mathcal{D}_{\acute{e}t}(X^{\diamond}, \mathbb{F}_\ell) \xrightarrow{b_X^*} \mathcal{D}_{\acute{e}t}(X^{\dagger}, \mathbb{F}_\ell).$$

When X is an affine scheme, fully-faithfulness of these functors is shown in [GL22, Lemma 4.1]. The passage to small v-stacks follows formally from this case.

Theorem 1.4 (Theorem 7.12)**.** *We have the following identification of small v-stacks*

$$\mathrm{Bun}_G^{\mathrm{mer}} \cong \mathfrak{B}(G)^{\dagger}$$

and an identification of maps $b^*_{\mathfrak{B}(G)} = \gamma^*$. A similar statement holds for the stack of local $\mathcal{G}$-shtukas for $\mathcal{G}$ a parahoric model of G.

Remark 1.5. This statement can be regarded as a version of Fargues' theorem in families (see Remark 7.13).

Corollary 1.6. *We have a fully-faithful comparison map*

$$\gamma^* \circ c^*\colon \mathcal{D}_{\acute{e}t}(\mathfrak{B}(G), \mathbb{F}_\ell) \to \mathcal{D}_{\acute{e}t}(\mathrm{Bun}_G^{\mathrm{mer}}, \mathbb{F}_\ell).$$

Theorem 1.4 provides an approach to prove that Ψ is fully-faithful. Indeed, it suffices to prove that $\sigma_!$ is fully-faithful when restricted to those objects in the essential image of $\gamma^* \circ c^*$. The advantage being that the geometry of $\mathrm{Bun}_G^{\mathrm{mer}}$ is much closer to that of Bun_G than that of $\mathfrak{B}(G)$.

Remark 1.7. We warn the reader that it is unknown to the authors whether $\mathcal{D}_{ét}(\mathfrak{B}(G), \mathbb{F}_\ell)$ agrees with $\mathrm{Shv}(\mathfrak{B}(G), \mathbb{F}_\ell)$ or not. There is a fully-faithful version of $\gamma^* \circ c^*$ for $\mathrm{Shv}(\mathfrak{B}(G), \mathbb{F}_\ell)$, but its target category is not $\mathcal{D}_{ét}(\mathrm{Bun}_G^{\mathrm{mer}}, \mathbb{F}_\ell)$. This, among other cohomological subtleties, will be addressed in subsequent work.

One of the main ingredient in our proof of Theorem 1.4 is the following statement.

Proposition 1.8 (Corollary 7.11)**.** *Every $\mathcal{G}$-bundle on the Fargues–Fontaine curve extends v-locally at ∞.*

This together with the main theorem of [Ans22] (which is itself an ingredient in the proof of Proposition 1.8) has as a consequence the classification of Corollary 1.9 below. We fix some notation. Let $S = \mathrm{Spa}(R, R^+)$ be a product of points with $R^\circ = \prod_{i\in I} O_{C_i}$ and a family of pseudo-uniformizers $\varpi_\infty = (\varpi_i)_{i\in I}$ such that ϖ_∞ defines the topology on R°. We moreover fix an untilt $S^\sharp$ given by a non-zero divisor $\xi_\infty = (\xi_i)_{i\in I}$. This induces for all $i \in I$ an untilt $C_i^\sharp$.

Corollary 1.9. *The following categories are equivalent:*

(1) The category of local shtukas over S with paw at $S^\sharp$.
(2) The category of Breuil–Kisin–Fargues modules over $\mathbb{A}_{\mathrm{inf}}(R^{\circ,\sharp})$.
(3) The category of I-indexed families $\{(M_i, \Phi_i)\}_{i\in I}$ of Breuil–Kisin–Fargues modules over $\mathbb{A}_{\mathrm{inf}}(O_{C_i^\sharp})$ with uniformly bounded poles and zeroes at ξ_∞.

Furthermore, a similar statement with $\mathcal{G}$-structure holds.

Remark 1.10. That vector bundles extend v-locally at ∞ was also shown independently by Zhang in her proof of Scholze's fiber product conjecture [Zha23, Proof of Proposition 8.14]. The previous version of this article discussed a proof of Proposition 1.8 that was substantially more complicated and had restrictions on the characteristic of E. It used the $\mathcal{M}_b$-charts of Fargues–Scholze to uniformize Bun_G and extend at ∞. The current approach uses the Beauville–Laszlo uniformization of Bun_G to extend at ∞. The argument provided in the current version of the article is quite close to the one of Zhang. We are very grateful to her for several conversations related to this.

1.3. **New proofs of two established results.** As a consequence of our considerations we found new approaches to previously proven theorems relating the geometry of $\mathfrak{B}(G)$ and Bun_G.

1.3.1. *The schematic comparison.* Recall the reduction functor introduced by the first author in [Gle24, §3]. Roughly, it is an analogue in the context of v-sheaves of the functor that takes a formal scheme to its reduced special fiber; the reduction X^{red} of a v-stack X on perfectoid spaces over $\mathbb{F}_p$ is a v-stack on perfect schemes over $\mathbb{F}_p$ (see also §2). The following theorem is a reformulation of a result of Pappas–Rapoport [PR24, Theorem 2.3.8], which in turn genealizes a result of Anschütz [Ans23, Theorem 1.1]. We take a new approach.

Theorem 1.11 (Theorem 7.14)**.** *We have an identification of scheme-theoretic v-stacks*

$$(\mathrm{Bun}_G)^{\mathrm{red}} \cong \mathfrak{B}(G)\,.$$

If $\mathcal{G}$ is a parahoric model of G, then a similar statement holds for the stack of $\mathcal{G}$-shtukas.

Remark 1.12. We regard Theorem 1.11 as a classicality statement. Anschütz proves the equivalence of categories $(\mathrm{Bun}_G)^{\mathrm{red}}(k) \cong \mathfrak{B}(G)(k)$ for an algebraically closed field $k/\mathbb{F}_p$ using the classification of vector bundles over the Fargues–Fontaine curve [Ans23, Theorem 3.4]. Pappas–Rapoport prove this more generally using the result of Anschütz and in particular rely on the φ-structure [PR24, Theorem 2.3.8]. We give a uniform proof and work directly with the category of v-vector bundles over $Y_{(0,\infty)}$ showing that classicality is unrelated to the φ-structure. Güthge also realized this independently [Güt23] (see Remark 3.22).

1.3.2. *The topological comparison.* Recall that $B(G)$ comes endowed with a topology induced by its partial order. We can also consider $B(G)^{\mathrm{op}}$ endowed with the topology induced by the opposite partial order. For either a schematic or an analytic v-stack X, let $|X|$ denote its underlying topological space. Viehmann [Vie23, Theorem 1.1] proves that $|\mathrm{Bun}_G|^{\mathrm{op}} \cong B(G)$. Rapoport–Richartz [RR96] and He [He16, Theorem 2.12] prove $|\mathfrak{B}(G)| \cong B(G)$.

We give an alternative proof of the following theorem.

Theorem 1.13 (Theorem 7.18)**.** *The natural maps are homeomorphisms*

$$|\mathrm{Bun}_G|^{\mathrm{op}} \cong |\mathfrak{B}(G)| \cong |\mathfrak{B}(G)^{\diamondsuit}| \,.$$

Remark 1.14. The proof of Theorem 1.13 relies on Proposition 1.8. In a previous version of this article, our approach to proving Proposition 1.8 implicitly relied on a version of Theorem 1.13 for GL_n. The current approach avoids this circularity.

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2. Notation, terminology and generalities

We fix the following notation throughout the text. Given a Huber pair (A, A^+), we let $\mathrm{Spa}(A, A^+)$ denote the preadic space of [SW20, §3.4]. Whenever $A^+ = A^\circ$ we simply write $\mathrm{Spa}\, A$. Analogously, given a Huber pair over $\mathbb{Z}_p$, we let $\mathrm{Spd}(A, A^+)$ denote the small v-sheaf of [Sch17, Lemma 15.1]. Whenever $A^+ = A^\circ$ we simply write $\mathrm{Spd}\, A$. We let Adic denote the category of analytic adic spaces over $\mathbb{Z}_p$ [SW20, Definition 3.2.1]. For an adic space X, by a geometric point of X, denoted by $\overline{x} \to X$, we mean a map of adic spaces $\overline{x} : \ \mathrm{Spa}(C, C^+) \to X$, where C is a complete non-Archimedean algebraically closed field and $C^+ \subseteq C$ is an open and bounded valuation subring. Whenever $C^+ = O_C$ we say that $\overline{x} \to X$ is geometric rank 1 point. If $\overline{x}$ is an isomorphism we will say that X (the space itself) is a geometric point.

As in the introduction, E denotes a non-Archimedean local field, we let $O_E \subseteq E$ denote the ring of integers, we let $\pi \in O_E$ denote a choice of uniformizer, we assume that $\mathbb{F}_q = O_E/\pi$, we denote by $\mathbb{C}$ a fixed completed algebraic closure of E.

We let $\mathrm{PSch}^{\mathrm{aff}}$ denote the category of perfect affine schemes over $\mathbb{F}_q$. If $\mathcal{S} = \mathrm{Spec}\,(A) \in \mathrm{PSch}^{\mathrm{aff}}$ with associated v-sheaf $\mathcal{S}^\diamond = \mathrm{Spd}(A, A)$, we denote by $\mathbb{W}A$ the topological ring of O_E-Witt vectors. More precisely, if E is of characteristic 0 then $\mathbb{W}A := \mathcal{W}(A) \otimes_{\mathbb{Z}_p} O_E$, where $\mathcal{W}(A)$ denotes the p-typical Witt vectors, and if E is of characteristic p then $\mathbb{W}A = A \hat{\otimes}_{\mathbb{F}_q} O_E \cong A[[\pi]]$. We let $Y_{\mathcal{S}^\diamond} := \mathrm{Spa}\,\mathbb{W}A[\frac{1}{\pi}]$, this is an analytic sous-perfectoid adic space (see Lemma 3.13). Note that $A \mapsto \mathbb{W}A$ is a functor, and hence $\mathcal{S} \mapsto Y_{\mathcal{S}^\diamond}$ also is. In particular, we have the functorial lift $\varphi : \ Y_{\mathcal{S}^\diamond} \to Y_{\mathcal{S}^\diamond}$ of the absolute Frobenius on $\mathcal{S}$.

We let $\mathrm{Perf}^{\mathrm{aff}}$ denote the category of affinoid perfectoid spaces over $\mathbb{F}_q$. Fix $S \in \mathrm{Perf}^{\mathrm{aff}}$. If $S = \mathrm{Spa}(R, R^+)$, we let $\mathbb{A}_{\mathrm{inf}}(R^+)$ denote $\mathbb{W}(R^+)$ endowed with the $(\pi, [\varpi])$-adic topology, where ϖ denotes a(ny) uniformizer of R^+. We let $\mathcal{Y}_S$ be the locus in $\mathrm{Spa}\,\mathbb{A}_{\mathrm{inf}}(R^+)$ where $[\varpi] \neq 0$ for some pseudo-uniformizer $\varpi \in R^+$, and Y_S be the locus in $\mathrm{Spa}\,\mathbb{A}_{\mathrm{inf}}(R^+)$ where $\pi \cdot [\varpi] \neq 0$. These are sous-perfectoid adic spaces [FS24, Proposition II.1.1]. Recall [SW20, §12.2] that after fixing a pseudo-uniformizer $\varpi \in R^+$ we have a continuous function

$$\kappa_\varpi : |\mathcal{Y}_S| \to [0, \infty]\,.$$

For every interval $I \subseteq [0, \infty]$ we let $\mathcal{Y}_{S,I} \subseteq \mathcal{Y}_S$ denote the interior of $\kappa_\varpi^{-1}(I)$. We use the notation $B_{S,I}$ to denote $\mathrm{H}^0(\mathcal{Y}_{S,I}, \mathcal{O})$ and $B^+_{S,I}$ to denote $\mathrm{H}^0(\mathcal{Y}_{S,I}, \mathcal{O}^+)$. If S is understood from the context we may drop it from the notation.

Remark 2.1. We want to point out that in this article we cite several references that discuss the geometry and the vector bundle theory of $\mathcal{Y}$ and related geometric objects. Occasionally, these sources have the working assumption that $E = \mathbb{Q}_p$ or that E is a characteristic 0 field. Nevertheless, the proofs often generalize to all non-Archimedean local fields E. For convenience of the reader, we have compiled a list of statements that we cite which although stated under restrictions on E hold more generally. [Ked20, Theorem 3.8], [PR24, Proposition 2.1.3], [RR96, Theorem 3.6 (ii)] and [Zha23, Proposition 11.16].

2.1. **Grothendieck topologies.** We endow Adic with the analytic topology. We endow $\mathrm{Perf}^{\mathrm{aff}}$ with the v-topology [Sch17, Definition 8.1]. We will consider several topologies on $\mathrm{PSch}^{\mathrm{aff}}$, mainly the scheme-theoretic v-topology [BS17, Definition 3.2], the arc-topology [BM21] and the proétale topology [BS15].

For the convenience of the reader we recall how these topologies are defined. Since we will work with sheaves of categories, we prefer to use the language of covering sieves [Lur09, §6.2.2]. All the topologies that we consider below are finitary. This means that if $X \in \mathrm{Perf}^{\mathrm{aff}}$ (or $X \in \mathrm{PSch}^{\mathrm{aff}}$) and $\mathcal{U} \subseteq \mathrm{Perf}^{\mathrm{aff}}_{/X}$ is a covering sieve, then there is a finite set of objects $\{Y_i \to X\}_{i=1}^n \subseteq \mathcal{U}$ such that the sieve generated by the Y_i is a covering sieve of X which refines $\mathcal{U}$. Since both $\mathrm{Perf}^{\mathrm{aff}}$ and $\mathrm{PSch}^{\mathrm{aff}}$ admit finite disjoint unions, every covering can be refined by the covering sieve generated by one map $Y \to X$. Therefore, it suffices to specify which maps of affinoid perfectoid spaces $\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(B, B^+)$ (or of affine schemes $\mathrm{Spec}\, A \to \mathrm{Spec}\, B$) are covers.

The v-topology on $\mathrm{Perf}^{\mathrm{aff}}$ declares every map $\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(B, B^+)$ to be a cover as long as $|\mathrm{Spa}(A, A^+)| \to |\mathrm{Spa}(B, B^+)|$ is surjective.

A map of affine schemes $\mathrm{Spec}\, A \to \mathrm{Spec}\, B$ is:

(1) A proétale cover if it is weakly étale in the sense of [BS15, Definition 1.2].
(2) A v-cover if for every map $B \to V$ with V a valuation ring there is an extension of valuation rings $V \subseteq W$ and a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}\, W & \longrightarrow & \mathrm{Spec}\, A \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, V & \longrightarrow & \mathrm{Spec}\, B\,. \end{array}$$

(3) An arc-cover if for every map $B \to V$ with V a rank 1 valuation ring there is an extension of rank 1 valuation rings $V \subseteq W$ and a commutative diagram as above.

We let Sets denote the category of sets, we let Grps denote the $(2,0)$-category of groupoids. We let Cat_1 be the $(2,1)$-category of small categories. We let $\mathrm{Cat}_1^{\otimes}$ (resp. $\mathrm{Cat}^{\otimes}_{1,O_E}$, $\mathrm{Cat}^{\otimes}_{1,E}$) denote the $(2,1)$-category of rigid symmetric monoidal small additive categories (resp. O_E-linear, E-linear symmetric monoidal small additive categories). Moreover, we let $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$, $\mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,O_E}$, $\mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E}$ denote the versions of $\mathrm{Cat}_1^{\otimes}$, $\mathrm{Cat}^{\otimes}_{1,O_E}$, $\mathrm{Cat}^{\otimes}_{1,E}$ equipped with exact structure. We clarify in Appendix A the precise meaning of these categories, and we show that they are presentable and compactly generated. The main purpose Appendix A is to justify that sheafification of presheaves with values in the above categories behaves as expected. The reader is encouraged to ignore the appendix and follow their intuition of what sheafification does and means.

In the body of the text we will be interested in extending the domain of definition of functors that originally are only defined over $\mathrm{Perf}^{\mathrm{aff}}$ (or $\mathrm{PSch}^{\mathrm{aff}}$). This is the case because many interesting spaces (like diamonds) are not members of $\mathrm{Perf}^{\mathrm{aff}}$, but can be regarded as functors over $\mathrm{Perf}^{\mathrm{aff}}$. For the convenience of the reader we recall how this can be done. We will need the following additional notation. We let Ani denote the $(\infty,0)$-category of anima and $\widehat{\mathrm{Cat}}_\infty$ the $(\infty,1)$-category of large categories. We identify the category of $(2,1)$-categories with the full subcategory of 2-truncated objects of $\widehat{\mathrm{Cat}}_\infty$, thus regarding $\mathrm{Cat}_1^{\otimes}$ and all of its versions above as objects in $\widehat{\mathrm{Cat}}_\infty$. In the following discussion category means ∞-category. Fix $\mathcal{T} \in \{\mathrm{Perf}^{\mathrm{aff}}, \mathrm{PSch}^{\mathrm{aff}}\}$.

Definition 2.2. Given a complete category C and a functor $\mathcal{F}\colon \mathcal{T}^{\mathrm{op}} \to C$, we will say that $\mathcal{F}$ is a C-valued v-sheaf (analogously C-valued proétale sheaf, schematic v-sheaf, and arc-sheaf) if for any $X \in \mathrm{Perf}^{\mathrm{aff}}$ and any covering sieve $S \subseteq \mathrm{Perf}^{\mathrm{aff}}_{/X}$ the limit map

$$\mathcal{F}(X) \to \varprojlim_{\{Y\to X\}\in S} \mathcal{F}(Y)$$

is an equivalence in C.

For a category C, we let $\mathcal{P}(\mathcal{T}, C)$ denote the category of C-valued presheaves, and if C is complete, we let $\mathcal{S}(\mathcal{T}, C) \subseteq \mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, C)$ be the full subcategory of C-valued v-sheaves as in Definition 2.2. If $C = \mathrm{Ani}$, we simply write $\mathcal{P}(\mathcal{T})$ and $\mathcal{S}(\mathcal{T})$. If $C = \mathrm{Grps}$, we write $\widetilde{\mathrm{Perf}}$ (resp. $\widetilde{\mathrm{PSch}}$) instead of $\mathcal{S}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Grps})$ (resp. $\mathcal{S}(\mathrm{PSch}^{\mathrm{aff}}, \mathrm{Grps})$). Every geometric object we consider lies either in $\widetilde{\mathrm{Perf}}$ or in $\widetilde{\mathrm{PSch}}$, and we refer to them as small v-stacks or small scheme-theoretic v-stacks.

Remark 2.3. Recall that the Yoneda embedding $y\colon \mathcal{T} \to \mathcal{P}(\mathcal{T})$ realizes $\mathcal{P}(\mathcal{T})$ as the $(\infty,1)$-category freely generated by $\mathcal{T}$ under small colimits. In particular, if C is a cocomplete $(\infty,1)$-category, then any functor $\mathcal{F}\colon \mathcal{T} \to C$ can uniquely be extended to a cocontinuous functor $\mathcal{G}\colon \mathcal{P}(\mathcal{T}) \to C$. By abuse of notation, we will still denote $\mathcal{G}$ by $\mathcal{F}$.

Remark 2.4. In the context of Remark 2.3, if C is presentable [Lur09, §5.5.1], the inclusion $\mathcal{S}(\mathcal{T}, C) \subseteq \mathcal{P}(\mathcal{T}, C)$ admits a left-adjoint [HM24, Lemma A.4.15.]

$$(-)^{\mathrm{sh}}\colon \mathcal{P}(\mathcal{T}, C) \to \mathcal{S}(\mathcal{T}, C)$$

which we can call *sheafification*. Furthermore, if $\mathcal{F} \in \mathcal{S}(\mathcal{T}, C)$ the natural cocontinuous map

$$\mathcal{F}\colon \mathcal{P}(\mathcal{T}) \to C$$

described above factors as a composition $\mathcal{P}(\mathcal{T}) \xrightarrow{\mathrm{sh}} \mathcal{S}(\mathcal{T}) \to C$.

Since $\widetilde{\mathrm{Perf}} \subseteq \mathcal{P}(\mathrm{Perf}^{\mathrm{aff}})$ is a full subcategory, given any functor $\mathcal{F}\colon \mathrm{Perf}^{\mathrm{aff}} \to C$ with C a cocomplete category, one can contemplate the value $\mathcal{F}(X) \in C$ for any $X \in \widetilde{\mathrm{Perf}}$ by Remark 2.3. An analogous statement holds for $X \in \widetilde{\mathrm{PSch}}$. This allows us to make several interesting constructions.

Definition 2.5. Let $\mathcal{F} \in \mathcal{P}(\mathrm{PSch}^{\mathrm{aff}}, C)$. We define

$$\mathcal{F}^{\diamond_{\mathrm{pre}}}, \mathcal{F}^{\dagger_{\mathrm{pre}}}, \mathcal{F}^{\Diamond_{\mathrm{pre}}} \in \mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, C)$$

with formula

(1) $\mathcal{F}^{\diamond_{\mathrm{pre}}}(R, R^+) := \mathcal{F}(\mathrm{Spec}\, R^+)$.
(2) $\mathcal{F}^{\dagger_{\mathrm{pre}}}(R, R^+) := \mathcal{F}(\mathrm{Spec}\, R^\circ)$.
(3) $\mathcal{F}^{\Diamond_{\mathrm{pre}}}(R, R^+) := \mathcal{F}(\mathrm{Spec}\, R)$.

If C is presentable, we define

$$\mathcal{F}^{\diamond}, \mathcal{F}^{\dagger}, \mathcal{F}^{\Diamond} \in \mathcal{S}(\mathrm{Perf}^{\mathrm{aff}}, C)$$

by applying sheafification to the functors above. We call these constructions *the small diamond*, *the dagger* and *the big diamond* constructions, respectively.

Remark 2.6. Since for every $\mathrm{Spa}(R, R^+)$ we have ring inclusions $R^+ \subseteq R^\circ \subseteq R$, we obtain morphisms in $\mathcal{S}(\mathrm{Perf}^{\mathrm{aff}}, C)$

$$\mathcal{F}^{\diamond} \to \mathcal{F}^{\dagger} \to \mathcal{F}^{\Diamond}.$$

Example 2.7. Given $S = \mathrm{Spec}\,(A) \in \mathrm{PSch}^{\mathrm{aff}}$, one can think of S via the Yoneda embedding as $S \in \widetilde{\mathrm{PSch}}$. Then one can verify that

$$S^{\diamond_{\mathrm{pre}}}(R, R^+) = \{A \to R^+ \mid \text{ring maps}\},$$
$$S^{\dagger_{\mathrm{pre}}}(R, R^+) = \{A \to R^\circ \mid \text{ring maps}\},$$
$$S^{\Diamond_{\mathrm{pre}}}(R, R^+) = \{A \to R \mid \text{ring maps}\}.$$

Moreover, by [Sch17, Theorem 8.7] these functors satisfy v-descent. Thus for all $? \in \{\diamond, \dagger, \Diamond\}$ one has $S^? = S^{?_{\mathrm{pre}}}$. Furthermore, one can see concretely that $S^\diamond$ is represented by $\mathrm{Spd}\, A$ when A is endowed with the discrete topology, and that $S^\Diamond$ is represented by $\mathrm{Spd}(A, \mathbb{F}_q^{\min})$, where $\mathbb{F}_q^{\min} \subseteq A$ is the integral closure of $\mathbb{F}_q$ in A. On the other hand, $S^\dagger$ is not representable, but it can be identified with a closed subsheaf of $S^\Diamond$ (the bounded locus, see [Gle24, Proposition 2.25, Definition 2.2]).

Definition 2.8. Let C be a complete category and let $\mathcal{F} \in \mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, C)$. We define

$$\mathcal{F}^{(\mathrm{red}_{\mathrm{pre}})} \in \mathcal{P}(\mathrm{PSch}^{\mathrm{aff}}, C)$$

by setting $\mathcal{F}^{(\mathrm{red}_{\mathrm{pre}})}(\mathrm{Spec}\, R) := \mathcal{F}(\mathrm{Spec}\, R^\diamond)$. Here we use Remark 2.3 to make sense of the right-side term. If C is presentable, we define

$$\mathcal{F}^{\mathrm{red}} \in \mathcal{S}(\mathrm{PSch}^{\mathrm{aff}}, C),$$

as the v-sheafification of $\mathcal{F}^{(\mathrm{red}_{\mathrm{pre}})}$.

Remark 2.9. As it turns out, the functor

$$\diamond : \mathrm{PSch}^{\mathrm{aff}} \to \widetilde{\mathrm{Perf}^{\mathrm{aff}}}$$

turns schematic v-covers into surjective maps of v-sheaves by [Gle24, Proposition 3.7]. For this reason, if $\mathcal{F} \in \mathcal{S}(\mathrm{Perf}^{\mathrm{aff}}, C)$ then $\mathcal{F}^{(\mathrm{red}_{\mathrm{pre}})} \in \mathcal{S}(\mathrm{PSch}^{\mathrm{aff}}, C)$, and $\mathcal{F}^{\mathrm{red}}(\mathrm{Spec}\, A) = \mathcal{F}(\mathrm{Spec}\, A^\diamond)$. This shows that $(-)^{\mathrm{red}} : \widetilde{\mathrm{Perf}} \to \widetilde{\mathrm{PSch}}$ is a right-adjoint to $\diamond : \widetilde{\mathrm{PSch}} \to \widetilde{\mathrm{Perf}}$.

We consider a final family of constructions. Given a topological ring R we let R_{disc} denote the ring R endowed with the discrete topology.

Definition 2.10. Let C be a complete category and let $\mathcal{F} \in \mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, C)$. We define, using Remark 2.3,

$$\mathcal{F}^{(\mathrm{mer}_{\mathrm{pre}})}, \mathcal{F}^{(\mathrm{An}_{\mathrm{pre}})} \in \mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, C)$$

with formulas

(1) $\mathcal{F}^{(\mathrm{mer}_{\mathrm{pre}})}(R, R^+) := \mathcal{F}(\mathrm{Spd}(R_{\mathrm{disc}}, R^+_{\mathrm{disc}}))$,
(2) $\mathcal{F}^{(\mathrm{An}_{\mathrm{pre}})}(R, R^+) := \mathcal{F}(\mathrm{Spd}(R_{\mathrm{disc}}, R_{\mathrm{disc}}))$.

If C is presentable we define

$$\mathcal{F}^{\mathrm{mer}}, \mathcal{F}^{\mathrm{An}} \in \mathcal{S}(\mathrm{Perf}^{\mathrm{aff}}, C)$$

as the v-sheafification of the functors defined above. When $C = \mathrm{Grps}$ we call

$$\mathcal{F}^{\mathrm{mer}}, \mathcal{F}^{\mathrm{An}} : \widetilde{\mathrm{Perf}} \to \widetilde{\mathrm{Perf}}$$

the *meromorphic* and the *unbounded* locus functors of $\mathcal{F}$.

Remark 2.11. For any affinoid perfectoid $\mathrm{Spa}(R, R^+) \in \mathrm{Perf}^{\mathrm{aff}}$ we have maps $(R, R^+) \leftarrow (R_{\mathrm{disc}}, R^+_{\mathrm{disc}}) \to (R_{\mathrm{disc}}, R_{\mathrm{disc}})$ of Huber pairs. Thus, if C is presentable, we obtain maps $\mathcal{F} \leftarrow \mathcal{F}^{\mathrm{mer}} \to \mathcal{F}^{\mathrm{An}}$. One can easily show that $\mathcal{F}^{\mathrm{An}}$ coincides with $(\mathcal{F}^{\mathrm{red}})^{\Diamond}$.

One of the main goals of this article is to better understand the correspondence

$$\begin{tikzcd} (\mathrm{Bun}_G)^{\mathrm{mer}} \arrow[r] \arrow[d] & \mathrm{Bun}_G \\ (\mathrm{Bun}_G)^{\mathrm{An}} & \end{tikzcd}$$

that one obtains from applying the considerations of Remark 2.11 to $\mathrm{Bun}_G \in \widetilde{\mathrm{Perf}}$.

2.2. **Combs and product of points.** The big advantage of working with the v-topology and the schematic v-topology is that plenty of questions can be reduced to studying valuation rings and their ultra-products.

Definition 2.12.
(1) We say that an affine scheme $S = \mathrm{Spec}\, A$ is a *comb* if for all $x \in \pi_0(S)$ the closed subscheme attached to x is of the form $\mathrm{Spec}\, V_x$, where V_x is a valuation ring with algebraically closed fraction field.
(2) We say that a comb is an *extremally disconnected comb* if $\pi_0(S)$ is an extremally disconnected Hausdorff space.
(3) If $A = \prod_{i \in I} V_i$, where each V_i is a valuation ring with algebraically closed fraction field, then we say that S is a *product comb*.

Remark 2.13. Observe that product combs are extremally disconnected combs. By [BS17, Lemma 6.2] any qcqs scheme admits a v-cover by a product comb, and extremally disconnected combs are w-contractible in the pro-étale topology, that is any pro-étale covering splits (this follows from the proof of [BS17, Lemma 6.2] and [BS15, Lemma 2.4.8]). The situation is similar to v-covers over $\mathrm{Perf}^{\mathrm{aff}}$, see Remark 2.15.

Definition 2.14. Suppose that $S = \mathrm{Spec}\, A \in \mathrm{PSch}^{\mathrm{aff}}$ is a product comb and $\varpi \in A$ is a non-zero divisor. Let $R^+ = \widehat{A}_{\varpi}$ be the ϖ-adic completion of A and let $R = R^+[\frac{1}{\varpi}]$. Then $\mathrm{Spa}(R, R^+)$ is a strictly totally disconnected space, and we call any space obtained this way a *product of points*.

Remark 2.15. It follows from [Gle24, Example 1.1] that products of points form a basis for the v-topology on $\mathrm{Perf}^{\mathrm{aff}}$. Moreover, any product of points is also w-contractible as a perfectoid space. Indeed, if (A, A^+) is a product of points, there is an open embedding $\mathrm{Spa}(A, A^+) \subseteq \mathrm{Spec}\,(A)$ which induces a bijection on closed points, since every maximal ideal in A is supported by a continuous valuation. Moreover, it induces a continuous bijection on path-components, which is hence a homeomorphism.

Lemma 2.16. *Let S be a product of points and $X \to S$ a pro-étale cover. Then, this morphism has a section $S \to X$.*

Proof. By Remark 2.15, S is w-contractible. Then the result follows from [MW23, Lemma 1.2]. □

Proposition 2.17. *Let $S = \mathrm{Spec}\, A$ be a product comb with $A = \prod_{i \in I} V_i$ and $\varpi \in A$ a non-zero divisor. Let $R^+ = \widehat{A}_{\varpi}$ be the ϖ-adic completion. Let ϖ_i be the image of ϖ in V_i which is also a non-zero divisor. Let $K_i^+ = \widehat{V}_{i,\varpi_i}$ be the ϖ_i-adic completion. Then, the family of projection maps $R^+ \to K_i^+$ induces a ring isomorphism $R^+ = \prod_{i \in I} K_i^+$.*

Proof. Let $I \times \mathbb{N}$ be the partial order with $(i_1, n_1) \leq (i_2, n_2)$ if $i_1 = i_2$ and $n_1 \leq n_2$. We have a functor from $I \times \mathbb{N}$ to the category of rings sending (i, n) to V_i/ϖ_i^n. The constructions of R^+ and $\prod_{i \in I} K_i^+$ correspond to two different ways of computing the limit of this diagram. □

Proposition 2.18. *If* $\mathrm{Spa}(R, R^+)$ *is a product of points, then* $\mathrm{Spec}\, R$ *is a comb.*

Proof. By Proposition 2.17, $R^+ = \prod_{i\in I} C_i^+$, where C_i^+ are valuation rings with algebraically closed fraction fields. Since ultraproducts of valuation rings with algebraically closed fraction field are again valuation rings with algebraically closed fraction fields, $\mathrm{Spec}\, R^+$ is a comb. Now, since affine Zariski localizations of combs are combs again, $\mathrm{Spec}\, R$ is a comb. □

We will use the following proposition implicitly throughout the article.

Proposition 2.19. *Let* H *be a locally profinite group. Then*

$$[*/\underline{H}]^{\lozenge} = [*/\underline{H}]^{\diamond} = [*/\underline{H}],$$

where $\underline{H}(S) = C^0(|S|, H)$ *parametrizes continuous maps from* $|S|$ *to* H.

Proof. Let $\mathrm{Spa}(R, R^+) \in \mathrm{Perf}^{\mathrm{aff}}$. Observe that $\underline{H}^{\diamond} = \underline{H}^{\lozenge} = \underline{H}$. Since $\lozenge$ (resp. $\diamond$) commutes with limits, it suffices to prove that the map $* \to [*/\underline{H}]^{\lozenge}$ (resp. $* \to [*/\underline{H}]^{\diamond}$) is surjective. This amounts to showing that if $\mathcal{F}$ is a $\underline{H}$-torsor for the schematic v-topology over $\mathrm{Spec}\, R$ (resp. $\mathrm{Spec}\, R^+$), then there is an analytic v-cover of $\mathrm{Spa}(R', R'^+) \to \mathrm{Spa}(R, R^+)$ such that $\mathcal{F}$ restricted to $\mathrm{Spec}\, R'$ is trivial. We can take $\mathrm{Spa}(R', R'^+)$ to be a product of points. It follows from a theorem of Gabber [HS21, Theorem 1.5] that every $\underline{H}$-torsor is pro-étale locally trivial. Since $\mathrm{Spec}\, R'$ (resp. $\mathrm{Spec}\, R'^+$) are extremally disconnected combs by Proposition 2.18, every pro-étale cover over them splits, see Remark 2.13. □

3. Categories of vector bundles

Our starting point is the functor

$$\mathcal{V} : (\mathrm{Adic})^{\mathrm{op}} \to \mathrm{Cat}_1^{\otimes,\mathrm{ex}}$$

which takes an analytic adic space to its category of vector bundles (i.e. locally free $\mathcal{O}_X$-modules). This is well-behaved, satisfies analytic descent, and satisfies that $\mathcal{V}(\mathrm{Spa}(R, R^+)) \simeq \mathrm{Proj}(R)$, where the latter denotes the symmetric monoidal additive category of finite dimensional projective R-modules [SW20, Theorem 5.2.8], [KL13, Theorem 2.7.7]. At this point we note that for a geometric point $\mathrm{Spa}(C, C^+)$ with canonical inclusion $\mathrm{Spa}(C, O_C) \subseteq \mathrm{Spa}(C, C^+)$ we obtain equivalences in $\mathrm{Cat}_1^{\otimes,\mathrm{ex}}$

$$\mathcal{V}(\mathrm{Spa}(C, C^+)) \simeq \mathcal{V}(\mathrm{Spa}(C, O_C)) \simeq \mathrm{Proj}(C).$$

In particular, for most of the statements that we discuss below one can replace the usage of geometric point by that of rank 1 geometric point.

Lemma 3.1. *Let* X *be a stably uniform analytic adic space and let* $\Sigma := [\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3]$ *be a sequence of vector bundles over* X*. The sequence* Σ *is exact if and only if for every geometric point* $\overline{x} \to X$ *the restricted sequence* $\overline{x}^*\Sigma$ *is exact.*

Proof. Since $\mathcal{V}$ is a sheaf, one can verify exactness locally and we may assume that $X = \mathrm{Spa}(R, R^+)$ and that each $\mathcal{E}_i$ is free. Since $\mathrm{Spa}(R, R^+)$ is uniform, it is reduced and we may verify that Σ is a complex on geometric points by [SW20, Theorem 5.2.1]. Injectivity on the left can be done similarly. It suffices to prove surjectivity, since exactness in the middle follows from this. We can reinterpret Σ as a sequence over $\mathrm{Spec}\, R$. Passing to determinant bundles, we may assume that $\mathcal{E}_3 = R$ and that the image of $\mathcal{E}_2$ generates an ideal $I \subseteq R$. If I is proper, then it is contained in a maximal ideal $I \subseteq \mathfrak{m} \subseteq R$. This shows that

$$\mathcal{E}_2 \otimes_R C(\mathfrak{m}) \to I \otimes_R C(\mathfrak{m}) \to 0 \to C(\mathfrak{m})$$

is not surjective and constructs a geometric point of $\mathrm{Spa}(R, R^+)$ for which $\overline{x}^*\Sigma$ is not exact. □

When we restrict the functor $\mathcal{V}$ to $\mathrm{Perf}^{\mathrm{aff}}$ one has stronger descent results. Indeed, by [SW20, Lemma 17.1.8], $\mathcal{V}: (\mathrm{Perf}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Cat}_1^{\otimes,\mathrm{ex}}$ is a v-sheaf. Moreover, by [SW20, Proposition 6.3.4] it makes sense to consider vector bundles on sous-perfectoid adic spaces. We consider functors

$$\mathcal{V}_{\mathcal{Y}}: (\mathrm{Perf}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Cat}_{1,O_E}^{\otimes,\mathrm{ex}} \quad \text{and} \quad \mathcal{V}_Y: (\mathrm{Perf}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Cat}_{1,E}^{\otimes,\mathrm{ex}}$$

with

$$\mathcal{V}_{\mathcal{Y}}(S) := \{\text{Vector bundles over } \mathcal{Y}_S\} \quad \text{and} \quad \mathcal{V}_Y(S) := \{\text{Vector bundles over } Y_S\}\,.$$

It follows from [SW20, Proposition 19.5.3] that both $\mathcal{V}_{\mathcal{Y}}$ and $\mathcal{V}_Y$ are v-sheaves. In this case, the exact structures on $\mathcal{V}_{\mathcal{Y}}(S)$ and $\mathcal{V}_Y(S)$ can be tested on geometric points of S by Lemma 3.1, so v-descent of the exact structure is immediate to verify.

The category of vector bundles on the Fargues–Fontaine curve [FS24, §II.2] can be defined by the pull-back square

$$\begin{tikzcd} \mathrm{Bun}_{\mathrm{FF}} \arrow[r] \arrow[d] & \mathcal{V}_Y \arrow[d, "\Delta"] \\ \mathcal{V}_Y \arrow[r, "(\mathrm{id},\varphi^*)"] & \mathcal{V}_Y \times \mathcal{V}_Y\,. \end{tikzcd}$$

In order to understand the functors $(\mathrm{Bun}_{\mathrm{FF}})^{\mathrm{mer}}$ and $(\mathrm{Bun}_{\mathrm{FF}})^{\mathrm{An}}$ giving rise to the correspondence

$$\begin{tikzcd} (\mathrm{Bun}_{\mathrm{FF}})^{\mathrm{mer}} \arrow[r] \arrow[d] & \mathrm{Bun}_{\mathrm{FF}} \\ (\mathrm{Bun}_{\mathrm{FF}})^{\mathrm{An}} & \end{tikzcd}$$

we will have to study the outcome of evaluating $\mathcal{V}_Y$ on v-sheaves of the form $\mathrm{Spd}(R_{\mathrm{disc}}, R_{\mathrm{disc}})$ and $\mathrm{Spd}(R_{\mathrm{disc}}, R^+_{\mathrm{disc}})$.

3.1. **E_∞-sous-perfectoid spaces.** We let $\mathrm{OpenSch} \subseteq \widetilde{\mathrm{Perf}}$ denote the full subcategory of v-sheaves which open locally admit an open immersion into a space of the form $S^\diamond$ for $S \in \mathrm{PSch}^{\mathrm{aff}}$. We note that by Remark 3.14, this full subcategory contains $\mathrm{Perf}^{\mathrm{aff}}$. The purpose of this section is, for $\mathcal{F} \in \mathrm{OpenSch}$, to construct the space $Y_{\mathcal{F}}$ and to prove Theorem 3.15 below, which characterizes $\mathcal{V}_Y(\mathcal{F})$.

The main observation is that the proof of [SW20, Proposition 19.5.3] can be generalized to a broad class of analytic adic spaces.

Definition 3.2. A Huber ring R over E is called E_∞-sousperfectoid if the completed tensor product $R\widehat{\otimes}_E E_\infty$ is perfectoid. If we have an adic space $X = \mathrm{Spa}(A, A^+)$ over $\mathrm{Spa}(E)$ such that A is E_∞-sousperfectoid, we call X *affinoid E_∞-sousperfectoid.*

Remark 3.3. Note that E_∞ is topologically countably generated over E and hence topologically free by [BGR84, §2.7, Theorem 4]. This shows that being E_∞-sousperfectoid implies being sousperfectoid [SW20, Definition 6.3.1]. Moreover, the proof of [SW20, Proposition 6.3.3.(1)] shows that being E_∞-sousperfectoid is stable under rational localization.

Definition 3.4. We define the category E_∞-SPerfd as the category of analytic adic spaces X over $\mathrm{Spa}(E)$ such that X can be covered by affinoid E_∞-sousperfectoid spaces.

Definition 3.5. A collection of morphisms $\{f_i : Y_i \to X\}_{i\in I}$ between E_∞-sousperfectoid spaces is a v-cover if for each quasicompact open subset $U \subseteq X$, there exists a finite subset $J \subseteq I$ and quasicompact open subsets $V_i \subseteq Y_i$ for $i \in J$, such that $U = \bigcup_{i\in J} f_i(V_i)$.

Proposition 3.6. *The category E_∞-SPerfd together with v-covers is a site.*

Proof. It suffices to show that fibre products exist, so let $Y \to X \leftarrow Z$ be a map of E_∞-sousperfectoid spaces. We can assume that $X = \mathrm{Spa}(A, A^+), Y = \mathrm{Spa}(B, B^+), Z = \mathrm{Spa}(C, C^+)$, where A, B, C are Tate and their completed tensor product with E_∞ over E is perfectoid. But then

$$(B\widehat{\otimes}_A C)\widehat{\otimes}_E E_\infty \cong (B\widehat{\otimes}_E E_\infty)\widehat{\otimes}_{A\widehat{\otimes}_E E_\infty}(C\widehat{\otimes}_E E_\infty)$$

and we know that the term on the right hand side is perfectoid since completed tensor products of perfectoid Tate Huber rings are perfectoid. This shows in particular that $B\widehat{\otimes}_A C$ is again E_∞-sousperfectoid and hence sheafy. □

Example 3.7. As it turns out, $\mathrm{Spa}(E_\infty)$ together with its natural map to $\mathrm{Spa}(E)$ is not an example of an E_∞-sousperfectoid space. Indeed, when $\mathrm{char}(E) = p$, this is evident since $\mathbb{F}_q(\!(t^{1/p^\infty})\!)\widehat{\otimes}_{\mathbb{F}_q(\!(t)\!)}\mathbb{F}_q(\!(t^{1/p^\infty})\!)$ is not perfect. When $\mathrm{char}(E) = 0$, we let $C := \widehat{\overline{E}}$ and $B := C\widehat{\otimes}_E C$ with ring of definition $B_0 := \mathcal{O}_C\widehat{\otimes}_{\mathcal{O}_E}\mathcal{O}_C$. Using that completed tensor products of perfectoid rings are again perfectoid twice, it suffices to show that B is not perfectoid by contradiction. For $n \in \mathbb{N}$, we fix a primitive p^n-th root of unity $\zeta_{p^n} \in C$. We now construct the elements

$$a_n := \sum_{i=0}^{p^n-1} \zeta_{p^n}^i \otimes \zeta_{p^n}^{-i} \in B.$$

We can now define elements $b_n := \frac{a_n}{p^{n/2}} \in B$ which are idempotent and hence power-bounded. However, $p^{(n-1)/2} \cdot b_n \notin B_0$ for all n such that $\zeta_{p^n} \notin E$ and hence for $n \gg 0$. This shows that B is not uniform, and consequently it is not perfectoid. The same proof shows that $\mathbb{C}_p$ is not E_∞-sousperfectoid.

We can generalize [SW20, Proposition 19.5.3] to perfect complexes. For an analytic adic space X, we denote by $\mathfrak{Perf}(X)$ the ∞-category of perfect complexes on X and by $\mathfrak{Perf}^{[a,b]}(X)$ its subcategory of complexes of tor-amplitude $[a, b]$. We note that the presheaves

$$\begin{aligned} \mathfrak{Perf} : \mathrm{Adic} &\to \mathrm{Cat}_\infty \\ X &\mapsto \mathfrak{Perf}(X) \end{aligned}$$

and

$$\begin{aligned} \mathfrak{Perf}^{[a,b]} : \mathrm{Adic} &\to \mathrm{Cat}_\infty \\ X &\mapsto \mathfrak{Perf}^{[a,b]}(X) \end{aligned}$$

satisfy analytic descent by [And21, Theorem 1.4].

Theorem 3.8. *The presheaves*

$$\begin{aligned} \mathfrak{Perf} : E_\infty\text{-SPerfd} &\to \mathrm{Cat}_\infty \\ X &\mapsto \mathfrak{Perf}(X) \end{aligned}$$

and

$$\begin{aligned} \mathfrak{Perf}^{[a,b]} : E_\infty\text{-SPerfd} &\to \mathrm{Cat}_\infty \\ X &\mapsto \mathfrak{Perf}^{[a,b]}(X) \end{aligned}$$

are sheaves for the v-topology.

Proof. The same proof method as in [AB21, Proposition 2.4] applies to this context. □

The category of vector bundles identifies with the full subcategory of perfect complexes which have tor-amplitude $[0, 0]$. Moreover, exactness, an E-linear structure and a $\otimes$-structure can be checked after passing to a v-cover so we get the following:

Corollary 3.9. *The association*

$$\begin{aligned} E_\infty\text{-SPerfd} &\to \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E} \\ X &\mapsto \mathcal{V}(X) \end{aligned}$$

is a sheaf for the v-topology.

Corollary 3.10. *The association*

$$\begin{aligned} E_\infty\text{-SPerfd} &\to \mathrm{Grps} \\ X &\mapsto \{G\text{-torsors over X}\} \end{aligned}$$

valued in Groupoids is a sheaf for the v-topology.

Corollary 3.11. *The site of E_∞-sousperfectoid spaces with the v-topology is subcanonical.*

Proof. We can use Theorem 3.8 and proceed analogously as in the second part of [Sch17, Corollary 8.6]. □

The main observation about E_∞-sousperfectoid spaces is that we can check a lot of properties of morphisms after base-changing to E_∞:

Proposition 3.12. *The functor*

$$\begin{aligned} E_\infty\text{-SPerfd} &\to \mathrm{Perfd}/\mathrm{Spa}(E_\infty) \\ X &\mapsto X \times_{\mathrm{Spa}(E)} \mathrm{Spa}(E_\infty) \end{aligned}$$

is conservative and faithful.

Proof. It suffices to show that the composition

$$E_\infty\text{-SPerfd} \to \mathrm{Perfd}/\mathrm{Spa}(E_\infty) \xrightarrow{(-)^\flat} \mathrm{Perf}/\mathrm{Spa}(E_\infty)^\flat$$

reflects isomorphisms. Let $f : Y \to X$ be a morphism in $E_\infty - \mathrm{SPerfd}$ such that

$$f_\infty : (Y \times_{\mathrm{Spa}(E)} \mathrm{Spa}(E_\infty))^\flat \to (X \times_{\mathrm{Spa}(E)} \mathrm{Spa}(E_\infty))^\flat$$

is an isomorphism. Then $f^\Diamond : Y^\Diamond \to X^\Diamond$ is an isomorphism, as this can be checked on v-covers of v-sheaves. Then, [Sch17, Lemma 15.6] tells us that $f : Y \to X$ is a homeomorphism. Observing that there are naturally split inclusions of sheaves $\mathcal{O}_X \hookrightarrow g_* \mathcal{O}_{X \times_{\mathrm{Spa}(E)} \mathrm{Spa}(E_\infty)}$ (and analogously for $\mathcal{O}_Y$), f is also an isomorphism on structure sheaves, which concludes the proof. We can proceed analogously to show that the functor is faithful. □

We now construct the spaces $Y_\mathcal{F}$.

Lemma 3.13. *There is, up to unique isomorphism, a pair $(Y_{(-)}, \cong)$, where $Y_{(-)}$ is a functor*

$$\begin{aligned} Y_{(-)} : \mathrm{OpenSch} &\to E_\infty\text{-SPerfd} \\ \mathcal{F} &\mapsto Y_\mathcal{F} \end{aligned}$$

such that:

(1) $Y_{\mathrm{Spec}\,(A)^\diamond} := \mathrm{Spa}(\mathbb{W}(A)[\frac{1}{\pi}], \mathbb{W}(A))$,
(2) $Y_{(-)}$ *preserves open immersions,*
(3) *and $\cong$ is a natural transformation of functors in* $\mathrm{Fun}(\mathrm{OpenSch}, \widetilde{\mathrm{Perf}})$

$$(Y_{(-)})^\Diamond \cong (-) \times \mathrm{Spd}(E)\,.$$

Furthermore, this construction commutes with products and more generally taking Čech nerves.

Proof. Assume that $\mathcal{S} = \mathrm{Spec}\,(A)$ is an affine perfect scheme. We claim that

$$Y_{\mathcal{S}^\diamond} = \mathrm{Spa}(\mathbb{W}(A)[\frac{1}{\pi}], \mathbb{W}(A))$$

is affinoid E_∞-sousperfectoid, where $\mathbb{W}(A)$ carries the π-adic topology. It suffices to show that

$$R = \mathbb{W}(A)[\frac{1}{\pi}]\widehat{\otimes}_E E_\infty \cong \mathbb{W}(A)[\frac{1}{\pi}]\widehat{\otimes}_{O_E} O_{E_\infty}$$

is perfectoid. Taking the open subring

$$R_0 = \mathbb{W}(A)\widehat{\otimes}_{O_E} O_{E_\infty},$$

we observe that $R_0/\pi^{1/p} \cong A \otimes_{\mathbb{F}_q} \mathbb{F}_q[t^{1/p^\infty}]/t^{1/p} \cong A[t^{1/p^\infty}]/t^{1/p}$ and $R_0/\pi \cong A \otimes_{\mathbb{F}_q} \mathbb{F}_q[t^{1/p^\infty}]/t \cong A[t^{1/p^\infty}]/t$. This implies that

$$\Phi\colon\ R_0/\pi^{1/p} \xrightarrow{\cong} R_0/\pi$$

is an isomorphism. Using [BMS18, Lemma 3.10 (ii)], we see that R_0 is integral perfectoid and [BMS18, Lemma 3.21] shows that $R_0[1/\pi] \cong R$ is perfectoid.

We now want to show that $(Y_{\mathcal{S}^\diamond})^\diamondsuit \cong \mathcal{S}^\diamond \times \mathrm{Spd}(E)$. Namely, we want to see that for any perfectoid space T over $\mathrm{Spa}(E)$, giving a map $T \to Y_{\mathcal{S}^\diamond}$ is equivalent to giving a map $T^\flat \to \mathcal{S}^\diamond = \mathrm{Spa}(A, A)$. We can assume that $T = \mathrm{Spa}(B, B^+)$ is affinoid. In this case, giving a map $T \to Y_{\mathcal{S}^\diamond}$ is the same as giving a map $\mathbb{W}(A) \to B^+$ (the image of π in B is always invertible since B lives over E). By the universal property of $\mathbb{W}(A)$, this is equivalent to giving a map $A \to B^+/\pi$ since B^+ is π-complete. Since A is perfect, this is in turn equivalent to giving a map $A \to (B^+)^\flat$, which is precisely a map $T^\flat \to \mathcal{S}^\diamond = \mathrm{Spa}(A, A)$.

We can now glue this to construct our desired space for general perfect schemes. Note that the small diamond functor $\diamond\colon\ \widetilde{\mathrm{PSch}^{\mathrm{aff}}} \to \widetilde{\mathrm{Perf}^{\mathrm{aff}}}$ preserves open subsheaves, which allows us to glue by using Proposition 3.12. For $\mathcal{F} \in \mathrm{Open}$ with an open immersion $\mathcal{F} \subseteq \mathcal{S}^\diamond$, we get an open immersion of v-sheaves $\mathcal{F} \times \mathrm{Spd}(E) \subseteq (Y_{\mathcal{S}^\diamond})^\diamondsuit$. It follows from [Sch17, Lemma 15.6] that there is a unique (necessarily E_∞-sousperfectoid) open subspace $Y_{\mathcal{F}} \subseteq Y_{\mathcal{S}^\diamond}$ inducing the open immersion $\mathcal{F} \times \mathrm{Spd}\, E \subseteq (Y_{\mathcal{S}^\diamond})^\diamondsuit$. Commutation with Čech nerves follows from Proposition 3.12. ☐

Remark 3.14. It is helpful to describe the spaces $Y_{\mathcal{F}}$ explicitly for different v-sheaves $\mathcal{F}$. Suppose that $X = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}^{\mathrm{aff}}$ and that $\mathcal{S} = \mathrm{Spec}\, R^+ \in \mathrm{PSch}^{\mathrm{aff}}$. Fix $\varpi \in R^+$ a pseudo-uniformizer.

(1) If $\mathcal{F} = \mathrm{Spd}(R^+, R^+)$, where R^+ carries the ϖ-adic topology, then we have an open immersion $\mathcal{F} \subseteq \mathcal{S}^\diamond$ corresponding to the locus where ϖ is topologically nilpotent (see [Gle24, Lemma 2.24]). Moreover, we get the E_∞-sousperfectoid space

$$Y_{\mathcal{F}} = Y_{X,(0,\infty]} = \mathrm{Spa}(\mathbb{A}_{\mathrm{inf}}(R^+), \mathbb{A}_{\mathrm{inf}}(R^+)) \setminus V(\pi).$$

In other words, $Y_{\mathcal{F}}$ is the curve with ∞ included.

(2) If $\mathcal{F} = \mathrm{Spd}(R_{\mathrm{disc}}, R^+_{\mathrm{disc}})$, where both rings carry the discrete topology, we have an open immersion $\mathcal{F} \subseteq \mathcal{S}^\diamond$ corresponding to the locus where $\varpi \neq 0$. Moreover, we get the E_∞-sousperfectoid space

$$Y_{\mathcal{F}} = \mathrm{Spa}(\mathbb{W}(R^+)[\frac{1}{\pi}], \mathbb{W}(R^+)) \setminus V([\varpi]).$$

(3) Finally, assume that $\mathcal{F} = X$. Then we have an open immersion $X \subseteq \mathcal{S}^\diamond$ corresponding to the intersection of the loci where $\varpi \neq 0$ and where ϖ is topologically nilpotent. We get the E_∞-sousperfectoid space

$$Y_X = \mathrm{Spa}(\mathbb{A}_{\mathrm{inf}}(R^+), \mathbb{A}_{\mathrm{inf}}(R^+)) \setminus V(\pi[\varpi])$$

from [FS24, Definition II.1.15], which satisfies that $Y_X^\diamondsuit \cong X \times \mathrm{Spd}(E)$. Using Proposition 3.12, we can construct the spaces Y_X for general perfectoid spaces X (over $\mathbb{F}_q$).

Theorem 3.15. *For any $\mathcal{F} \in \mathrm{OpenSch}$ there is a functorial identification*

$$\mathcal{V}_Y(\mathcal{F}) \simeq \mathcal{V}(Y_{\mathcal{F}})\,.$$

Proof. By definition, $\mathcal{V}_Y(\mathcal{F}) = \varprojlim \mathcal{V}(Y_X)$ as X varies in $\mathrm{Perf}^{\mathrm{aff}}/\mathcal{F}$. By Lemma 3.13, we have a functor

$$Y_{(-)} : \mathrm{OpenSch} \to E_\infty\text{-SPerfd}$$

that preserves v-covers. By Corollary 3.9, $\mathcal{V}(Y_{(-)})$ satisfies v-descent. Moreover, $\mathrm{Perf}^{\mathrm{aff}} \subseteq \mathrm{OpenSch}$ is a basis for the v-topology, so $\mathrm{Perf}^{\mathrm{aff}}/\mathcal{F}$ is a covering sieve. Thus we obtain

$$\mathcal{V}_Y(\mathcal{F}) = \varprojlim \mathcal{V}(Y_X) \cong \mathcal{V}(Y_{(\varprojlim X)}) = \mathcal{V}(Y_{\mathcal{F}})\,.$$

□

Remark 3.16. The above proof has already found an application in [ABM24] in which a 6-functor formalism $D_{(0,\infty)}(-)$ for solid quasi-coherent sheaves on the spaces $Y_{(-)}$ is constructed. Adjusting the above proof to this setting gives an analagous result $D_{(0,\infty)}(\mathrm{Spd}\,\mathbb{F}_p) \cong D_{\square}(\mathrm{AnSpec}\,\mathbb{Q}_p)$, see [ABM24, Theorem 6.3.1].

Let $X_S := Y_S/\varphi^{\mathbb{Z}}$ denote the relative Fargues–Fontaine curve with respect to a perfectoid space S from [FS24, Definition II.1.15]. We define the presheaves of ∞-categories

$$\begin{aligned} \mathfrak{Perf}_Y : (\mathrm{Perf}^{\mathrm{aff}})^{\mathrm{op}} &\to \mathrm{Cat}_\infty \\ S &\mapsto \mathfrak{Perf}(Y_S) \end{aligned}$$

and

$$\begin{aligned} \mathfrak{Perf}_{\mathrm{FF}} : (\mathrm{Perf}^{\mathrm{aff}})^{\mathrm{op}} &\to \mathrm{Cat}_\infty \\ S &\mapsto \mathfrak{Perf}(X_S) \end{aligned}$$

which are sheaves by Theorem 3.8. Now note that the proof of Theorem 3.15 generalizes to perfect complexes.

Theorem 3.17. *For any $\mathcal{F} \in \mathrm{OpenSch}$ there is a functorial identification of ∞-categories*

$$\mathfrak{Perf}_Y(\mathcal{F}) \simeq \mathfrak{Perf}(Y_{\mathcal{F}})\,.$$

In particular, this positively answers [Ans23, Conjecture 1.2]. The third author wants to thank Johannes Anschütz for related discussions.

Definition 3.18. Let $S \in \mathrm{PSch}^{\mathrm{aff}}$. We define the category $\mathfrak{Perf}_{\mathfrak{B}}(S)$ of perfect complexes of isocrystals over S as the equalizer of ∞-categories

$$\mathfrak{Perf}_{\mathfrak{B}}(S) \longrightarrow \mathfrak{Perf}(Y_{S^\diamond}) \underset{\varphi^*}{\overset{\mathrm{id}}{\rightrightarrows}} \mathfrak{Perf}(Y_{S^\diamond})\,.$$

We call a pair $(K, \alpha_K : K \cong \varphi^* K)$ in $\mathfrak{Perf}_{\mathfrak{B}}(S)$ strictly perfect if it can be written as a finite limit of objects $(\mathcal{E}, \alpha_{\mathcal{E}} : \mathcal{E} \cong \varphi^*\mathcal{E})$, where $\mathcal{E}$ is a vector bundle on $Y_{S^\diamond}$. This is equivalent to (K, α_K) representing an honest bounded complex of isocrystals on $Y_{S^\diamond}$.

Theorem 3.19. *Let $S \in \mathrm{PSch}^{\mathrm{aff}}$. The natural functor of ∞-categories*

$$\mathfrak{Perf}_{\mathfrak{B}}(S) \to \mathfrak{Perf}_{\mathrm{FF}}(S^\diamond)$$

is an equivalence. Moreover, the category $\mathfrak{Perf}_{\mathfrak{B}}(S)$ is equivalent to the category of bounded complexes of isocrystals.

Proof. We note that for all $S \in \mathrm{Perf}^{\mathrm{aff}}$ there is an equalizer diagram of ∞-categories

$$\mathfrak{Perf}_{\mathrm{FF}}(S) \longrightarrow \mathfrak{Perf}(Y_S) \underset{\varphi^*}{\overset{\mathrm{id}}{\rightrightarrows}} \mathfrak{Perf}(Y_S). \tag{3.1}$$

The equivalence of categories then follows from Theorem 3.17 applied to $\mathcal{F} = \mathcal{S}^\diamond$. The second statement follows from [AB21, Proposition 2.7] applied to the equalizer diagram of ∞-categories

$$\mathfrak{Perf}_{\mathfrak{B}}(\mathcal{S}) \longrightarrow \mathfrak{Perf}(Y_{\mathcal{S}^\diamond}) \underset{\varphi^*}{\overset{\mathrm{id}}{\rightrightarrows}} \mathfrak{Perf}(Y_{\mathcal{S}^\diamond}).$$

$\square$

As a consequence, we get:

Corollary 3.20. *Let $\lambda \in \mathbb{Q}^\times$ with associated absolute Banach–Colmez spaces $\mathcal{BC}_{\lambda,i} := \mathcal{BC}(\mathcal{O}(\lambda)[i])$ for $i \in \{0,1\}$*[1]*. Then $\mathcal{BC}^{\mathrm{red}}_{\lambda,i} = 0$, and the counit map $(\mathcal{BC}^{\mathrm{red}}_{\lambda,i})^\diamond \to \mathcal{BC}_{\lambda,i}$ corresponds to the zero section.*

Proof. Let $\mathcal{S} = \mathrm{Spec}\,(A) \in \mathrm{PSch}^{\mathrm{aff}}$. By Theorem 3.17,

$$\mathcal{BC}_{\lambda,i}(\mathcal{S}^\diamond) = H^i(R\Gamma_{\mathfrak{Perf}_{\mathfrak{B}}(\mathrm{Spec}\,(A))}(\mathcal{O}(\lambda))) = 0,$$

where $\mathcal{O}(\lambda)$ denotes the simple standard isocrystal[2] over $\mathrm{Spec}\,(A)$ of slope λ. A direct computation shows that

$$R\Gamma_{\mathfrak{Perf}_{\mathfrak{B}}(\mathrm{Spec}\,(A))}(\mathcal{O}(\lambda)) = [\mathcal{O}(\lambda) \xrightarrow{\varphi_{\mathcal{O}(\lambda)} - \mathrm{Id}} \mathcal{O}(\lambda)] \overset{\lambda \neq 0}{\simeq} 0,$$

where $\varphi_{\mathcal{O}(\lambda)} : \mathcal{O}(\lambda) \to \mathcal{O}(\lambda)$ denotes the φ-linear automorphism of $\mathcal{O}(\lambda)$. $\square$

3.2. **On $\mathcal{V}^{\mathrm{mer}}_Y$ and $\mathcal{V}^{\mathrm{An}}_Y$.** In this subsection we analyze $\mathcal{V}^{\mathrm{mer}}_Y$ and $\mathcal{V}^{\mathrm{An}}_Y$, the emphasis will be on clarifying their structure as objects in $\mathcal{S}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes}_{1,E})$. The main point is that both of these objects can be approximated by separated presheaves Definition A.3 and that these are easier to understand.

Definition 3.21. Consider the functors

$$\mathcal{V}_{\mathbb{W}} \in \mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,O_E}) \quad \text{and} \quad \mathcal{V}^{\mathrm{sch}}_{\mathbb{W}} \in \mathcal{P}(\mathrm{PSch}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,O_E})$$

with

$$\mathcal{V}_{\mathbb{W}}(\mathrm{Spa}(R,R^+)) := \{\text{Finite projective modules over } \mathbb{W}R\} =: \mathcal{V}^{\mathrm{sch}}_{\mathbb{W}}(\mathrm{Spec}\,R).$$

From [SW20, Corollary 17.1.9] (applied to the case where $n = \infty$ and $R^\sharp = R$) it follows that $\mathcal{V}_{\mathbb{W}}$ is a v-sheaf. From [BS17, Theorem 4.1.(ii)] it follows that $\mathcal{V}^{\mathrm{sch}}_{\mathbb{W}}$ is a scheme-theoretic v-sheaf.

Remark 3.22. It is also clear that $\mathcal{V}_{\mathbb{W}} \simeq (\mathcal{V}^{\mathrm{sch}}_{\mathbb{W}})^{(\Diamond_{\mathrm{pre}})} \simeq (\mathcal{V}^{\mathrm{sch}}_{\mathbb{W}})^{\Diamond}$. We expect that the identity $\mathcal{V}^{\mathrm{sch}}_{\mathbb{W}} \simeq (\mathcal{V}_{\mathbb{W}})^{\mathrm{red}}$ also holds. When E is of mixed-characteristic this latter identity is a result of Güthge [Güt23, § 3].

Definition 3.23. Consider functors

$$\mathcal{V}_{\mathbb{W}}[\frac{1}{\pi}], \mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}] \in \mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes}_{1,E}) \quad \text{and} \quad \mathcal{V}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}] \in \mathcal{P}(\mathrm{PSch}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes}_{1,E})$$

given by the formulas

$$\mathcal{V}_{\mathbb{W}}[\frac{1}{\pi}] := \mathcal{V}_{\mathbb{W}} \otimes_{O_E} E,\ \mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}] := \mathcal{V}_{\mathcal{Y}} \otimes_{O_E} E \text{ and } \mathcal{V}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}] := \mathcal{V}^{\mathrm{sch}}_{\mathbb{W}} \otimes_{O_E} E.$$

---

[1] Here, we use the notation from [FS24, Section II.2]

[2] Here, we use the notation from Definition 5.1

Remark 3.24. At the moment, we do not endow $\mathcal{V}_{\mathbb{W}}[\frac{1}{\pi}]$, $\mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}]$ or $\mathcal{V}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}]$ with exact structure, but later we will identify these categories with other categories that carry a natural exact structure.

Lemma 3.25. *Let $\mathcal{T} \in \{\mathrm{Perf}^{\mathrm{aff}}, \mathrm{PSch}^{\mathrm{aff}}\}$. Given $\mathcal{C} \in \mathcal{S}(\mathcal{T}, \mathrm{Cat}^{\otimes}_{1,O_E})$ be a v-sheaf of O_E-linear categories, let $\mathcal{D} := \mathcal{C} \otimes_{O_E} E$. Then $\mathcal{D} \in \mathcal{P}(\mathcal{T}, \mathrm{Cat}^{\otimes}_{1,E})$ is a separated presheaf.*

Proof. By Proposition A.4 and Lemma A.6, it suffices to show that for every v-cover $[U \to X] \in \mathrm{Perf}^{\mathrm{aff}}$ the map

$$\mathcal{D}(X) \to \mathrm{Desc.}(\mathcal{D}, X/U)$$

is fully faithful, or equivalently that for any two objects the presheaf of morphisms is a v-sheaf. By construction, objects in $\mathcal{D}(X)$ agree with objects in $\mathcal{C}(X)$. Since objects in $\mathcal{C}(X)$ are dualizable, the diagramatic characterization of dualizable objects implies that objects in $\mathcal{D}(X)$ are also dualizable. In particular, we may compute morphisms in terms of the $\otimes$-unit and internal Hom-objects. Indeed, for $V, W \in \mathcal{D}(X)$ we have $\mathrm{Hom}(V, W) \simeq \mathrm{Hom}(\mathbb{1} \otimes V, W) \simeq \mathrm{Hom}(\mathbb{1}, V^\vee \otimes W)$. It remains to show that for all objects $V \in \mathcal{C}(X)$ the presheaf of E-vector spaces

$$\mathcal{H} := \mathrm{Hom}_{\mathcal{C}\otimes_{O_E} E}(\mathbb{1}, V) = \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, V) \otimes_{O_E} E$$

is a sheaf. Nevertheless, $\mathcal{H}$ can be written as a sequential colimit of the form

$$\mathcal{H} = \varinjlim\, [\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, V) \xrightarrow{\cdot\pi} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, V) \xrightarrow{\cdot\pi} \dots].$$

Since in the category of O_E-modules finite limits commute with filtered colimits, and $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, V)$ is a sheaf of O_E-modules, we can conclude that $\mathcal{H}$ is also a sheaf of E-vector spaces. □

Corollary 3.26. *$\mathcal{V}_{\mathbb{W}}[\frac{1}{\pi}]$, $\mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}]$ and $\mathcal{V}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}]$ are separated presheaves.*

Definition 3.27. Consider the functor

$$\mathcal{V}^{\mathrm{sch}}_{Y} \in \mathcal{P}(\mathrm{PSch}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E}) \text{ with}$$

$$\mathcal{V}^{\mathrm{sch}}_{Y}(\mathrm{Spec}\, R) := \{\text{Finite projective modules over } \mathbb{W}R[\frac{1}{\pi}]\}.$$

Proposition 3.28. *The following statements hold:*

(1) $\mathcal{V}^{\mathrm{sch}}_{Y} \simeq (\mathcal{V}_{Y})^{\mathrm{red}}$.
(2) *$\mathcal{V}^{\mathrm{sch}}_{Y}$ is an arc-sheaf.*
(3) *If $S = \mathrm{Spec}\, R$ is a comb then $\mathcal{V}^{\mathrm{sch}}_{Y}(S) \simeq \mathcal{V}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}](S)$ in $\mathrm{Cat}^{\otimes}_{1,E}$.*
(4) *The v-sheafification of $\mathcal{V}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}]$ is equivalent to $\mathcal{V}^{\mathrm{sch}}_{Y}$ in $\mathcal{S}(\mathrm{PSch}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes}_{1,E})$.*

Proof. The first statement follows from Theorem 3.15 and the definitions. By [Iva23, Proposition 5.9] $\mathcal{V}^{\mathrm{sch}}_{Y}$ is an arc-sheaf. By [Iva23, Theorem 6.1] the values of $\mathcal{V}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}]$ and $\mathcal{V}^{\mathrm{sch}}_{Y}$ agree on combs. Since combs form a basis for the schematic v-topology (see Remark 2.13), the fourth claim follows. □

We see that $\mathcal{V}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}]$ has a natural immersion into $\mathcal{V}^{\mathrm{sch}}_{Y}$ and acquires the structure of a separated presheaf with values in $\mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E}$ since it inherits an exact structure.

Proposition 3.29. *We have the following identities:*

(1) $(\mathcal{V}^{\mathrm{sch}}_{Y})^{\Diamond_{\mathrm{pre}}} \simeq (\mathcal{V}_{Y})^{(\mathrm{An}_{\mathrm{pre}})}$ *and* $\mathcal{V}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}]^{\Diamond_{\mathrm{pre}}} \simeq \mathcal{V}_{\mathbb{W}}[\frac{1}{\pi}]$ *in* $\mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E})$.
(2) $\mathcal{V}_{\mathbb{W}}[\frac{1}{\pi}]^{\mathrm{sh}} \simeq \mathcal{V}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}]^{\Diamond} \simeq (\mathcal{V}^{\mathrm{sch}}_{Y})^{\Diamond} \simeq \mathcal{V}^{\mathrm{An}}_{Y}$ *in* $\mathcal{S}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes}_{1,E})$.

Proof. The first claim follows directly from Theorem 3.15 and the definitions. The second claim follows from Proposition 3.28 and Proposition 2.18. □

Corollary 3.30. *The presheaf $\mathcal{V}_{\mathbb{W}}[\frac{1}{\pi}]$ (with the exact structure inherited from $(\mathcal{V}_Y^{\mathrm{sch}})^{\lozenge_{\mathrm{pre}}}$) is separated and sheafifies to $\mathcal{V}_Y^{\mathrm{An}}$.*

Proof. This follows from Lemma A.6. □

We now move on to rewrite $(\mathcal{V}_Y)^{\mathrm{mer}}$.

Proposition 3.31. *We have a Cartesian square in $\mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes}_{1,E})$*

$$\begin{tikzcd} \mathcal{V}_{\mathcal{Y}} \arrow[r] \arrow[d] & \mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}] \arrow[d] \\ \mathcal{V}_{\mathbb{W}} \arrow[r] & \mathcal{V}_{\mathbb{W}}[\frac{1}{\pi}]. \end{tikzcd}$$

Proof. The argument is a standard application of Beauville–Laszlo descent [SW20, Lemma 5.2.9]. We provide the details for the convenience of the reader. Fix $S \in \mathrm{Perf}^{\mathrm{aff}}$. By analytic descent, there is a Cartesian diagram

$$\begin{tikzcd} \mathcal{V}_{\mathcal{Y}}(S) \arrow[r] \arrow[d] & \mathcal{V}(Y_{[1,\infty),S}) \arrow[d] \\ \mathcal{V}(Y_{[0,1],S}) \arrow[r] & \mathcal{V}(Y_{[1,1],S}). \end{tikzcd}$$

These spaces correspond to the loci $Y_{[0,1],S} = \{|\pi| \leq |[\varpi]| \neq 0\}$ and $Y_{[1,\infty),S} = \{|[\varpi]| \leq |\pi| \neq 0\}$. Moreover, $\mathcal{V}(Y_{[0,1],S})$ is the category of finite projective modules over $B_{[0,1],S}$. Since π is already invertible in $Y_{[1,\infty),S}$ and $Y_{[1,1],S}$, this formally leads to the following commutative diagram with Cartesian squares

$$\begin{tikzcd} \mathcal{V}_{\mathcal{Y}} \arrow[r] \arrow[d] & \mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}](S) \arrow[r] \arrow[d] & \mathcal{V}(Y_{[1,\infty),S}) \arrow[d] \\ (\mathrm{Proj}\, B_{[0,1],S}) \arrow[r] & (\mathrm{Proj}\, B_{[0,1],S}) \otimes_{O_E} E \arrow[r] & \mathcal{V}(Y_{[1,1],S}). \end{tikzcd}$$

Furthermore, $\mathcal{V}_{\mathbb{W}}(S)$ is equivalent to $\mathrm{Proj}(\mathbb{W}R)$. Since we have the identity of rings $\mathbb{W}R = \widehat{(B_{[0,1]})}_{\pi}$, we have the following Cartesian diagram by [SW20, Lemma 5.2.9].

$$\begin{tikzcd} \mathrm{Proj}\, B_{[0,1],S} \arrow[r] \arrow[d] & \mathcal{V}_{\mathbb{W}}(S) \arrow[d] \\ (\mathrm{Proj}\, B_{[0,1],S}) \otimes_{O_E} E \arrow[r] & \mathcal{V}_{\mathbb{W}}[\frac{1}{\pi}](S). \end{tikzcd}$$

This implies that the commutative diagram below is also Cartesian

$$\begin{tikzcd} \mathcal{V}_{\mathcal{Y}}(S) \arrow[r] \arrow[d] & \mathrm{Proj}\, B_{[0,1],S} \arrow[r] \arrow[d] & \mathcal{V}_{\mathbb{W}}(S) \arrow[d] \\ \mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}](S) \arrow[r] & (\mathrm{Proj}\, B_{[0,1],S}) \otimes_{O_E} E \arrow[r] & \mathcal{V}_{\mathbb{W}}[\frac{1}{\pi}](S). \end{tikzcd}$$

□

Proposition 3.32. *The diagrams*

$$\begin{tikzcd}
\mathcal{V}_{\mathcal{Y}} \arrow[r] \arrow[d] & \mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}] \arrow[r] \arrow[d] & (\mathcal{V}_Y)^{\mathrm{mer}_{\mathrm{pre}}} \arrow[d] \\
\mathcal{V}_{\mathbb{W}} \arrow[r] & \mathcal{V}_{\mathbb{W}}[\frac{1}{\pi}] \arrow[r] & (\mathcal{V}_Y)^{(\mathrm{An}_{\mathrm{pre}})}
\end{tikzcd}$$

are Cartesian in $\mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes}_{1,O_E})$. *Moreover, the right square is Cartesian in* $\mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes}_{1,E})$ *and the outer square is Cartesian in* $\mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,O_E})$.

Proof. Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}^{\mathrm{aff}}$ and let $T = \mathrm{Spd}(R_{\mathrm{disc}}, R^+_{\mathrm{disc}})$. Pick a pseudo-uniformizer $\varpi \in R^+$. As defined in Section 2, we denote the two different topologies on the ring of O_E-Witt vectors by $\mathbb{W}(R^+)$ and $\mathbb{A}_{\mathrm{inf}}(R^+)$.

The category $\mathcal{V}_{\mathcal{Y}}(S)$ is the category of vector bundles over $\mathrm{Spa}(\mathbb{A}_{\mathrm{inf}}(R^+))_{\{[\varpi]\neq 0\}}$. Arguing as in Proposition 3.31, we get the following Cartesian diagram

$$\begin{tikzcd}
\mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}](S) \arrow[r] \arrow[d] & (\mathrm{Proj}\, B_{[0,1],S}) \otimes_{O_E} E \arrow[d] \\
\mathcal{V}(Y_{[1,\infty),S}) \arrow[r] & \mathcal{V}(Y_{[1,1],S}) .
\end{tikzcd}$$

On the other hand, $(\mathcal{V}_Y)^{(\mathrm{mer}_{\mathrm{pre}})}(S) \simeq \mathcal{V}(Y_T)$. This is the category of vector bundles on $\mathrm{Spa}\,\mathbb{W}(R^+)_{\{\pi\cdot[\varpi]\neq 0\}}$. This space is the union of the loci $\{|[\varpi]| \leq |\pi| \neq 0\}$ and $\{|\pi| \leq |[\varpi]| \neq 0\}$. The former agrees with $Y_{[1,\infty),S}$ while the latter is affinoid of the form $\mathrm{Spa}(B^{\mathrm{disc}}_{[0,1]}[\frac{1}{\pi}], B^{\mathrm{disc},+}_{[0,1]}[\frac{1}{\pi}])$. By analytic descent, we have a pullback diagram

$$\begin{tikzcd}
\mathcal{V}(Y_T) \arrow[r] \arrow[d] & (\mathrm{Proj}\, B^{\mathrm{disc}}_{[0,1],S}[\frac{1}{\pi}]) \arrow[d] \\
\mathcal{V}(Y_{[1,\infty),S}) \arrow[r] & \mathcal{V}(Y_{[1,1],S}) .
\end{tikzcd}$$

Note that the natural map $B^{\mathrm{disc}}_{[0,1]} \to B_{[0,1]}$ is a continuous isomorphism of rings (which is not a homeomorphism!), see [SW20, Lemma 14.3.1]. Hence, we have an identification $(\mathrm{Proj}\, B^{\mathrm{disc}}_{[0,1],S}) \simeq (\mathrm{Proj}\, B_{[0,1],S})$ in $\mathrm{Cat}^{\otimes}_{1,O_E}$. Together with Proposition 3.31 and Beauville–Laszlo descent (see [SW20, Lemma 5.2.9]), this leads to the following commutative diagram with Cartesian squares

$$\begin{tikzcd}
\mathcal{V}_{\mathcal{Y}}(S) \arrow[r] \arrow[d] & \mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}](S) \arrow[r] \arrow[d] & \mathcal{V}(Y_T) \arrow[r] \arrow[d] & \mathcal{V}(Y_{[1,\infty),S}) \arrow[d] \\
(\mathrm{Proj}\, B_{[0,1],S}) \arrow[r] \arrow[d] & (\mathrm{Proj}\, B_{[0,1],S}) \otimes_{O_E} E \arrow[r] \arrow[d] & (\mathrm{Proj}\, B^{\mathrm{disc}}_{[0,1],S}[\frac{1}{\pi}]) \arrow[r] \arrow[d] & \mathcal{V}(Y_{[1,1],S}) \\
(\mathrm{Proj}\, \mathbb{W}(R)) \arrow[r] & (\mathrm{Proj}\, \mathbb{W}(R)) \otimes_{O_E} E \arrow[r] & (\mathrm{Proj}\, \mathbb{W}(R)[\frac{1}{\pi}]) .
\end{tikzcd}$$

It follows from the definitions that $(\mathrm{Proj}\, \mathbb{W}(R)) \simeq \mathcal{V}_{\mathbb{W}}(S)$, $(\mathrm{Proj}\, \mathbb{W}(R)) \otimes_{O_E} E \simeq \mathcal{V}_{\mathbb{W}}(S)[\frac{1}{\pi}]$, and $(\mathrm{Proj}\, \mathbb{W}(R)[\frac{1}{\pi}]) \simeq (\mathcal{V}_Y)^{(\mathrm{An}_{\mathrm{pre}})}$ which allow us to conclude. $\square$

Corollary 3.33. *The square*

$$\begin{tikzcd}
\mathcal{V}_{\mathcal{Y}} \arrow[r] \arrow[d] & (\mathcal{V}_Y)^{\mathrm{mer}} \arrow[d] \\
\mathcal{V}_{\mathbb{W}} \arrow[r] & (\mathcal{V}_Y)^{\mathrm{An}}
\end{tikzcd}$$

is Cartesian in $\mathcal{S}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,O_E})$.

Corollary 3.34. *The following statements hold:*

(1) *The natural map* $\mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}] \to (\mathcal{V}_Y)^{(\mathrm{mer}_{\mathrm{pre}})}$ *is fully-faithful.*

(2) *The v-sheafification of* $\mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}]$ *is* $(\mathcal{V}_Y)^{\mathrm{mer}}$.

Proof. The first statement follows from the proof of Proposition 3.32 and from the fact that

$$(\mathrm{Proj}\,\mathbb{W}(R)) \otimes_{O_E} E \to (\mathrm{Proj}\,\mathbb{W}(R)[\frac{1}{\pi}])$$

is fully-faithful.

To show $\mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}]^{\mathrm{sh}} \simeq \mathcal{V}_Y^{\mathrm{mer}}$ it suffices by Remark 2.15 to show that $\mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}](S) \simeq (\mathcal{V}_Y)^{(\mathrm{mer}_{\mathrm{pre}})}(S)$ when S is a product of points. If $\mathrm{Spa}(R, R^+)$ is a product of points, $(\mathrm{Proj}\,\mathbb{W}(R)) \otimes_{O_E} E \simeq (\mathrm{Proj}\,\mathbb{W}(R)[\frac{1}{\pi}])$ by [Iva23, Theorem 6.1]. Using Proposition 3.32, we can conclude. □

Corollary 3.35. *The presheaf* $\mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}]$ *(with the exact structure inherited from* $(\mathcal{V}_Y)^{\mathrm{mer}_{\mathrm{pre}}}$*) is separated and sheafifies to* $\mathcal{V}_Y^{\mathrm{An}}$.

Proof. This follows from Lemma A.6. □

Remark 3.36. Although for $S \in \mathrm{Perf}^{\mathrm{aff}}$ the categories $\mathcal{V}_Y^{\mathrm{An}}(S) \in \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E}$ and $\mathcal{V}_Y^{\mathrm{mer}}(S) \in \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E}$ together with their exact structure are fairly abstract, one can still have some formal control over them by applying Lemma A.7 to $\mathcal{V}_{\mathbb{W}}[\frac{1}{\pi}]$ and $\mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}]$. Indeed, these presheaves are separated by Corollary 3.35 and Corollary 3.30.

4. Meromorphic vector bundles on the Fargues–Fontaine curve

In this section we study $\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}$ and $\mathrm{Bun}^{\mathrm{An}}_{\mathrm{FF}}$. We note that their structure as objects in $\mathcal{S}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes}_{1,E})$ is easy to deduce from our study of $\mathcal{V}^{\mathrm{mer}}_{\mathcal{Y}}$ and $\mathcal{V}^{\mathrm{An}}_{\mathcal{Y}}$ simply by adding Frobenius structure. Nevertheless, as we will see, the Frobenius structure allows us to further understand their behavior as objects in $\mathcal{S}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E})$.

4.1. Dieudonné modules and Isocrystals.

For this subsection we fix a test object $S = \mathrm{Spec}\, A \in \mathrm{PSch}^{\mathrm{aff}}$.

Definition 4.1. A *Dieudonné module* over S is a pair $(\mathcal{E}, \Phi_{\mathcal{E}})$, where $\mathcal{E}$ is a finite projective module over $\mathbb{W}(A)$ and $\Phi_{\mathcal{E}}$ is an isomorphism

$$\Phi_{\mathcal{E}} : \varphi^* \mathcal{E}_{Y_{S^\diamond}} \to \mathcal{E}_{Y_{S^\diamond}} .$$

An *isocrystal* over S is a pair $(\mathcal{F}, \Phi_{\mathcal{F}})$, where $\mathcal{F}$ is a vector bundle over $Y_{S^\diamond}$ and $\Phi_{\mathcal{F}}$ is an isomorphism

$$\Phi_{\mathcal{F}} : \varphi^* \mathcal{F} \to \mathcal{F} .$$

Morphisms of these data are φ-equivariant maps. We declare a sequence of morphisms to be exact if the underlying sequence of projective modules (resp. vector bundles) is exact. We denote these categories by $\mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}}(S) \in \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,O_E}$ and $\mathfrak{B}(S) \in \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E}$. These rules organize into functors $\mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}} \in \mathcal{P}(\mathrm{PSch}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,O_E})$ and $\mathfrak{B} \in \mathcal{P}(\mathrm{PSch}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E})$.

Remark 4.2. Dieudonné modules are also considered by Pappas–Rapoport under the name *meromorphic Frobenius crystals* [PR24, Definition 2.3.6].

Proposition 4.3. *We have Cartesian diagrams*

$$\begin{tikzcd} \mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}} \arrow[r] \arrow[d] & \mathcal{V}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}] \arrow[r] \arrow[d, "\Delta"] & \mathcal{V}^{\mathrm{sch}}_{Y} \arrow[d, "\Delta"] \\ \mathcal{V}^{\mathrm{sch}}_{\mathbb{W}} \arrow[r, "(\mathrm{id},\varphi^*)"] & \mathcal{V}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}] \times \mathcal{V}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}] \arrow[r] & \mathcal{V}^{\mathrm{sch}}_{Y} \times \mathcal{V}^{\mathrm{sch}}_{Y} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{B} \arrow[r] \arrow[d] & \mathcal{V}^{\mathrm{sch}}_{Y} \arrow[d, "\Delta"] \\ \mathcal{V}^{\mathrm{sch}}_{Y} \arrow[r, "(\mathrm{id},\varphi^*)"] & \mathcal{V}^{\mathrm{sch}}_{Y} \times \mathcal{V}^{\mathrm{sch}}_{Y} . \end{tikzcd}$$

Proof. These are formal reinterpretations of Definition 4.1. □

Definition 4.4. We define $\mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}] \in \mathcal{P}(\mathrm{PSch}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes}_{1,O_E})$ by requiring that the following is a Cartesian diagram

$$\begin{tikzcd} \mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}] \arrow[r] \arrow[d] & \mathcal{V}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}] \arrow[d, "\Delta"] \\ \mathcal{V}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}] \arrow[r, "(\mathrm{id},\varphi^*)"] & \mathcal{V}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}] \times \mathcal{V}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}] . \end{tikzcd}$$

One can think of $\mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}](S)$ as the category of isocrystals over S that admit a lattice.

Proposition 4.5. *The following statements hold.*

(1) $\mathfrak{B} \in \mathcal{S}(\mathrm{PSch}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E})$ *and* $\mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}} \in \mathcal{S}(\mathrm{PSch}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,O_E})$.

(2) $\mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}] \in \mathcal{P}(\mathrm{PSch}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes}_{1,O_E})$ *is v-separated and* $\mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}]^{\mathrm{sh}} \simeq \mathfrak{B}$.

(3) The following diagram has Cartesian squares

$$\begin{tikzcd} \mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}} \arrow[r] \arrow[d] & \mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}] \arrow[r] \arrow[d] & \mathfrak{B} \arrow[d] \\ \mathcal{V}^{\mathrm{sch}}_{\mathbb{W}} \arrow[r] & \mathcal{V}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}] \arrow[r] & \mathcal{V}^{\mathrm{sch}}_{Y} . \end{tikzcd}$$

Proof. We note that the property of being a separated presheaf (resp. a sheaf) is stable under finite limits of presheaves by Proposition A.5. Then, by Proposition 3.28 the first two claims hold. The last claim follows formally from Definition 4.4, Proposition 4.3 and the fact that $\mathcal{V}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}]$ is separated with $\mathcal{V}^{\mathrm{sch}}[\frac{1}{\pi}]^{\mathrm{sh}} \simeq \mathcal{V}^{\mathrm{sch}}_{Y}$ □

Proposition 4.6. *Let* $\Sigma := [\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3]$ *be a sequence in* $\mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}}(S)$. *Then* Σ *is exact if and only if for every geometric point* $\bar{x} \to S$, *the sequence* $\Sigma_{\bar{x}}$ *is exact. If* Σ *is already a complex then it suffices to check exactness on geometric points with closed image in* S.

Proof. Since $\mathcal{V}^{\mathrm{sch}}_{\mathbb{W}}$ is a Zariski sheaf and since a basis over $\mathrm{Spec}\, A$ always deforms to a basis over $\mathrm{Spec}\, \mathbb{W}A$, we may assume that each $\mathcal{E}_i$ is free and of constant finite rank. The maps $\mathcal{E}_i \to \mathcal{E}_j$ are now given by matrices with values in $\mathbb{W}A = A^{\mathbb{N}}$ (as sets). The map $A \to \prod_{\bar{x} \to S} C_{\bar{x}}$ is injective, so one can check on geometric points that the sequence is a complex.

Once we know the sequence is a complex, exactness can be checked on closed points of $\mathrm{Spec}\, \mathbb{W}A$. As we vary over geometric points $\overline{x} \to S$ with closed image, the induced family of maps $\mathbb{W}(\overline{x})\colon \mathrm{Spec}\, \mathbb{W}C \to \mathrm{Spec}\, \mathbb{W}A$ covers all closed points of $\mathrm{Spec}\, \mathbb{W}A$. This allows us to conclude. □

To prove an analogue of Proposition 4.6 for $\mathfrak{B}$, we first give a reinterpretation.

Proposition 4.7. *The following statements hold.*

(1) For all $S \in \mathrm{PSch}^{\mathrm{aff}}$, $\mathrm{Bun}_{\mathrm{FF}}(S^{\diamond}) \simeq \mathfrak{B}(S)$ *in* $\mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E}$.

(2) $\mathfrak{B} \simeq (\mathrm{Bun}_{\mathrm{FF}})^{\mathrm{red}}$ *in* $\mathcal{S}(\mathrm{PSch}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E})$.

Proof. Both claims follow formally from the definitions and from Proposition 3.28. □

Remark 4.8. The result of Proposition 4.7 is implicitly proved during the proof of [PR24, Theorem 2.3.8] when E is of mixed characteristic. Their approach relied on Sen theory which is a tool that is only available in mixed characteristic. A previous version of this article also relied on Sen theory and for this reason we also had to restrict to the mixed characteristic setup.

Proposition 4.9. *Let* $S = \mathrm{Spec}\, R$ *and* $\Sigma := [\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3] \in \mathfrak{B}(S)$ *be a sequence such that the underlying projective modules* $\mathcal{E}_i$ *over* $\mathbb{W}R[\frac{1}{\pi}]$ *have constant rank* $\mathrm{rk}.(\mathcal{E}_i) = r_i$ *and* $r_1 + r_3 = r_2$. *The following statements hold.*

(1) *The sequence is exact if and only if for every geometric point* $\overline{x} \to S$ *the sequence* $\Sigma_{\overline{x}} \in \mathfrak{B}(\overline{x})$ *is exact.*
(2) *Moreover, if the sequence is already assumed to be a complex, then exactness can be checked on geometric points* $\overline{x} \to S$ *whose image is a closed point.*

Proof. The forward implication is evident. Assume that for every geometric point of S, the sequence is exact. By Proposition 4.5 we may test exactness v-locally. Thus, we may assume that $S = \mathrm{Spec}\, R$ is a comb and by [Iva23, Theorem 6.1] that all the underlying projective modules are free. We write $M_1 \in \mathrm{M}_{r_2\times r_1}(\mathbb{W}(R)[\frac{1}{\pi}])$ and $M_2 \in \mathrm{M}_{r_3\times r_2}(\mathbb{W}(R)[\frac{1}{\pi}])$ the matrices representing the maps $\mathcal{E}_1 \to \mathcal{E}_2$ and $\mathcal{E}_2 \to \mathcal{E}_3$, respectively. The induced map $\mathcal{E}_1 \to \mathcal{E}_3$ is the 0 map if and only if the matrix $M_2 \cdot M_1 = 0$. This can be tested on geometric points since R is perfect and in particular reduced. Exactness can now be expressed in terms of the rank of M_1 and M_2 at the different points of $\mathrm{Spa}\, \mathbb{W}(R)[\frac{1}{\pi}]$.

The locus where M_1 has rank strictly smaller to r_1 is a Zariski closed subset (cut out by the minors of M_1) $Z \subseteq \mathrm{Spa}\, \mathbb{W}(R)[\frac{1}{\pi}]$. Moreover, since the map $\mathcal{E}_1 \to \mathcal{E}_2$ is φ-equivariant, we have $\varphi(Z) = Z$. Indeed, the rank of M_1 equals the rank of $\varphi(M_1)$.

Suppose $Z \neq \emptyset$ and let $z \in Z$. Endow R with the discrete topology and consider the projection map $f:\ \mathrm{Spd}\, \mathbb{W}(R)[\frac{1}{\pi}] \to \mathrm{Spd}(R)$. By the classification of points in the olivine spectrum either $f(z)$ is d-analytic or it is discrete, see [Gle24, Definition 2.2]. In the following, we will argue that if $Z \neq \emptyset$ then there is $z \in Z$ such that $f(z)$ is algebraic in the sense of [Gle24, Definition 2.2.(1)]. Consider the quotient $g:\ \mathrm{Spd}\, \mathbb{W}(R)[\frac{1}{\pi}] \to \mathrm{Spd}\, \mathbb{W}(R)[\frac{1}{\pi}]/\varphi$. By φ-invariance, Z has the form $g^{-1}(Z')$ for a closed subset $Z' \subseteq |\mathrm{Spd}\, \mathbb{W}(R)[\frac{1}{\pi}]/\varphi|$. Recall from [FS24, Definition II.1.19] the diamond Div^1_E, which parametrizes degree 1 divisors on the Fargues–Fontaine curve. We have an identification at the level of topological spaces

$$\alpha:\ |\mathrm{Spd}\, \mathbb{W}(R)[\frac{1}{\pi}]/\varphi| \cong |\mathrm{Spd}(R) \times (\mathrm{Div}^1_E)|$$

which fits in the following commutative diagram.

$$\begin{tikzcd}
{|\mathrm{Spd}\, R \times \mathrm{Spd}\, E|} \arrow[r] \arrow[d] \arrow[dd, bend right=60, "|f|"'] & {|(\mathrm{Spd}\, R \times \mathrm{Spd}\, E)/\mathrm{Frob}_R \times \mathrm{id}_E|} \arrow[dl, "\alpha"] \arrow[dd] \\
{|\mathrm{Spd}\, R \times \mathrm{Div}^1_E|} \arrow[d] & \\
{|\mathrm{Spd}\, R|} \arrow[r, "\simeq"] & {|\mathrm{Spd}\, R/\mathrm{Frob}_R|}\ .
\end{tikzcd}$$

This shows that $f(Z)$ is the image of Z' under the projection map $|\mathrm{Spd}\, R \times \mathrm{Div}^1_E| \to |\mathrm{Spd}\, R|$.

By [FS24, Proposition II.1.21], this image is a closed subset of $\mathrm{Spd}\, R$. Furthermore, since Z is stable under vertical generization, the same is true about $f(Z)$. If $z \in f(Z)$ and it is discrete, then its largest vertical generization is already algebraic. Suppose instead that $f(z)$ is a d-analytic point. Recall that every point d-analytic point in $\mathrm{Spd}(R, R)$ is formal in the sense of [Gle24, Definition 2.2.(4)], and that formal points have a unique formal specialization [Gle24, Proposition 2.9.(2)]. Since $f(Z) \subseteq \mathrm{Spd}(R, R)$ is closed, it must contain the formal specialization of $f(z)$. In particular, we see that $f(Z)$ has a discrete point and by the above an algebraic one as we wanted to show.

We showed above that $f(Z)$ contains an algebraic point. We fix $x \in f(Z)$ that is algebraic, in particular there is a unique $y \in \mathrm{Spec}\,(R)$ inducing x. If $k(y)$ is the residue field of y we obtain a map $\mathrm{Spd}\, k(y) \to \mathrm{Spd}\, R$, and the only point of $|\mathrm{Spd}\, R|$ in the image is x. In this case $f^{-1}(f(x)) = \mathrm{Spd}\, k(y) \times \mathrm{Spd}\, E = \mathrm{Spd}\, \mathbb{W}(k(y))[\frac{1}{\pi}]$, which consists of one point. In particular, $\mathrm{Spd}\, \mathbb{W}(k(y))[\frac{1}{\pi}] \cap Z \neq \emptyset$ implies $\mathrm{Spa}\, \mathbb{W}(k(y))[\frac{1}{\pi}] \subseteq Z$. Since $\mathbb{W}(k(y))[\frac{1}{\pi}]$ is a field, this shows that every r_1-minor in M_1 thought of as an element in $(R^{\mathbb{Z}})^{r_2 \cdot r_1} \supseteq \mathrm{M}_{r_2 \times r_1}(\mathbb{W}(R)[\frac{1}{\pi}])$ vanishes identically when restricted to $k(y)^{\mathbb{Z}}$. The same must be true for every point in the Zariski closure of $\mathrm{Spec}\,(k(y)) \subseteq \mathrm{Spec}\,(R)$. In particular, we have found a closed point $\overline{x} \to \mathrm{Spec}\,(R)$ for which $\mathcal{E}_{1,\overline{x}} \to \mathcal{E}_{2,\overline{x}}$ is not injective. This contradicts our assumption, so $Z = \emptyset$. A similar argument proves that $\mathcal{E}_2 \to \mathcal{E}_3$ is surjective and by rank considerations the sequence is also exact in the middle. □

We use the $\diamondsuit$-functor to consider the categories of Dieudonné modules and of isocrystals as analytic objects.

Definition 4.10. If $S = \mathrm{Spa}(R, R^+)$ we let $\mathrm{Sht}_{\mathbb{W}}(S) := (\mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}})^{\diamondsuit_{\mathrm{pre}}}(S) = \mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}}(\mathrm{Spec}\, R)$. This rule defines a functor $\mathrm{Sht}_{\mathbb{W}} \in \mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,O_E})$.

Proposition 4.11. *The following statements hold.*

(1) $\mathrm{Sht}_{\mathbb{W}} \in \mathcal{S}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,O_E})$.

(2) We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Sht}_{\mathbb{W}} \arrow[r] \arrow[d] & \mathcal{V}_{\mathbb{W}}[\frac{1}{\pi}] \arrow[r] \arrow[d, "\Delta"] & (\mathcal{V}_Y)^{\mathrm{An}} \arrow[d, "\Delta"] \\
\mathcal{V}_{\mathbb{W}} \arrow[r, "(\mathrm{id},\varphi^*)"] & \mathcal{V}_{\mathbb{W}}[\frac{1}{\pi}] \times \mathcal{V}_{\mathbb{W}}[\frac{1}{\pi}] \arrow[r] & (\mathcal{V}_Y)^{\mathrm{An}} \times (\mathcal{V}_Y)^{\mathrm{An}}
\end{tikzcd}$$

with Cartesian squares.

Proof. This formally follows from Proposition 4.3, Proposition 3.29, Corollary 3.26 and Remark 3.22. □

Definition 4.12. Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}^{\mathrm{aff}}$. We let $\mathrm{Sht}_{\mathbb{W}}[\frac{1}{\pi}](S) := \mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}](\mathrm{Spec}\, R)$ and $\mathfrak{B}^{\diamondsuit_{\mathrm{pre}}}(S) = \mathfrak{B}(\mathrm{Spec}\, R)$. These rules organize into objects $\mathrm{Sht}_{\mathbb{W}}[\frac{1}{\pi}] \in \mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes}_{1,E})$ and $\mathfrak{B}^{\diamondsuit_{\mathrm{pre}}} \in \mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E})$. We let $\mathfrak{B}^{\diamondsuit}$ be the v-sheafification of $\mathfrak{B}^{\diamondsuit_{\mathrm{pre}}}$. We call objects of $\mathfrak{B}^{\diamondsuit}(S)$ *analytic isocrystals* over S.

Remark 4.13. It follows from Remark 2.11 and Proposition 4.7 that $(\mathrm{Bun}_{\mathrm{FF}})^{\mathrm{An}} \simeq \mathfrak{B}^{\diamondsuit}$.

Proposition 4.14. *Let* $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}^{\mathrm{aff}}$*, the following hold:*

(1) $\mathrm{Sht}_{\mathbb{W}}[\frac{1}{\pi}]$ *is a v-separated presheaf in* $\mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes}_{1,E})$.

(2) The v-sheafification of $\mathrm{Sht}_{\mathbb{W}}[\frac{1}{\pi}]$ *is equivalent to* $\mathfrak{B}^{\diamondsuit}$ *in* $\mathcal{S}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes}_{1,E})$.

(3) A sequence in $\mathrm{Sht}_{\mathbb{W}}[\frac{1}{\pi}]$ *is exact in* $\mathfrak{B}^{\diamondsuit}$ *if and only if it is exact in* $\mathfrak{B}^{\diamondsuit_{\mathrm{pre}}}$.

Proof. The first two claims follow formally from Corollary 3.26 and Proposition 3.29.

For the third claim let $\Sigma := [\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3] \in \mathrm{Sht}_{\mathbb{W}}[\frac{1}{\pi}](S)$ be a sequence i.e. Σ is a sequence in $\mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}](\mathrm{Spec}\, R)$. Since $\mathfrak{B}^{\diamondsuit_{\mathrm{pre}}}(S) \to \mathfrak{B}^{\diamondsuit}(S)$ is a map in $\mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E}$, it is clear that if Σ is exact in $\mathfrak{B}^{\diamondsuit_{\mathrm{pre}}}$

then it is also exact in $\mathfrak{B}^\diamondsuit$. Assume that Σ is exact in $\mathfrak{B}^\diamondsuit(S)$. By definition, this means that Σ is exact in $\mathfrak{B}(\operatorname{Spec} R')$ for a v-cover $\operatorname{Spa}(R', R'^+) \to \operatorname{Spa}(R, R^+)$. Since $\mathrm{Sht}_{\mathbb{W}}[\frac{1}{\pi}] \to \mathfrak{B}^\diamondsuit$ is fully-faithful, we deduce that the sequence is a complex. By the second part of Proposition 4.9, we can check exactness on closed points of $\operatorname{Spec} R$. Since $\operatorname{Spa}(R', R'^+) \to \operatorname{Spa}(R, R^+)$ is a v-cover, every closed point of $\operatorname{Spec} R$ is in the image of $\operatorname{Spec} R' \to \operatorname{Spec} R$. Indeed, closed points of $\operatorname{Spec} R$ support continuous valuations each of which will lift to a continuous valuation of R'. We have shown that if a sequence becomes exact in $\mathfrak{B}(\operatorname{Spec} R')$, then it was already exact in $\mathfrak{B}(\operatorname{Spec} R)$. □

Proposition 4.15. *Let $S = \operatorname{Spa}(R, R^+)$. Let $\Sigma = [\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3]$ be a sequence with $\mathcal{E}_i \in \mathfrak{B}^\diamondsuit(S)$, and each of constant rank $\mathrm{rk}.(\mathcal{E}_i) = r_i$ such that $r_1 + r_3 = r_2$. The sequence is exact if and only if for every geometric point $\overline{x} \to S$ the sequence $\overline{x}^*\Sigma$ with $\overline{x}^*\mathcal{E}_i \in \mathfrak{B}^\diamondsuit(\overline{x})$ is exact.*

Proof. The forward implication is evident. Assume that for every geometric point of S the sequence is exact. By the definition of the exact structure on $\mathfrak{B}^\diamondsuit(S)$ via sheafification, we may test exactness v-locally. More precisely, it suffices to find a v-cover $f : S' \to S$ with $S' = \operatorname{Spa}(R', R'^+)$ such that each $\mathcal{E}_i \in \mathrm{Sht}_{\mathbb{W}}[\frac{1}{\pi}](S')$ and such that $g^*\Sigma$ is exact in $\mathfrak{B}^{\diamondsuit_{\mathrm{pre}}}(S')$. Without loss of generality, $S = S'$ and $\mathcal{E}_i \in \mathfrak{B}^{\diamondsuit_{\mathrm{pre}}}(S)$. Since the map $R \to \prod_{x \in \operatorname{Spa}(R,R^+)} C_x$ is injective, we can test on geometric points if the map is a complex. Once we know it is a complex, by Proposition 4.9 we can test exactness on closed points of $\operatorname{Spec} R$. As every closed point of $\operatorname{Spec} R$ supports a geometric point of $\operatorname{Spa}(R, R^+)$, the proof is complete. □

4.2. **Shtukas, isoshtukas and meromorphic vector bundles.**

Definition 4.16. Let $S = \operatorname{Spa}(R, R^+) \in \mathrm{Perf}^{\mathrm{aff}}$. A *crystalline shtuka* over S is a pair $(\mathcal{E}, \Phi_{\mathcal{E}})$, where $\mathcal{E}$ is a vector bundle over $\mathcal{Y}_S$ and $\Phi_{\mathcal{E}}$ is an isomorphism

$$\Phi_{\mathcal{E}} : (\varphi^*\mathcal{E})_{Y_S} \to \mathcal{E}_{Y_S}$$

that is meromorphic (cf. [SW20, Definition 5.3.5]) along $\pi = 0$. Morphisms of these data are φ-equivariant maps. We declare a sequence of morphisms to be exact if the underlying maps of vector bundles form an exact sequence. We denote this category by $\mathrm{Sht}_{\mathcal{Y}}(S) \in \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,O_E}$. This induces a functor $\mathrm{Sht}_{\mathcal{Y}} \in \mathcal{P}(\mathrm{PSch}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,O_E})$.

Proposition 4.17. *The following statements hold.*

(1) $\mathrm{Sht}_{\mathcal{Y}} \in \mathcal{S}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,O_E})$.

(2) We have the following commutative diagram with Cartesian squares:

$$\begin{tikzcd}
\mathrm{Sht}_{\mathcal{Y}} \arrow[r] \arrow[d] & \mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}] \arrow[r] \arrow[d, "\Delta"] & (\mathcal{V}_Y)^{\mathrm{mer}} \arrow[d, "\Delta"] \\
\mathcal{V}_{\mathcal{Y}} \arrow[r, "(\mathrm{id},\varphi^*)"] & \mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}] \times \mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}] \arrow[r] & (\mathcal{V}_Y)^{\mathrm{mer}} \times (\mathcal{V}_Y)^{\mathrm{mer}}
\end{tikzcd}$$

Proof. That the left-hand square is Cartesian is a reinterpretation of Definition 4.16. That the right-hand square is Cartesian follows formally from Corollary 3.34 and Corollary 3.26. □

Definition 4.18. We define $\mathrm{Sht}_{\mathcal{Y}}[\frac{1}{\pi}] \in \mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes}_{1,E})$ by requiring that the following is a Cartesian diagram

$$\begin{tikzcd}
\mathrm{Sht}_{\mathcal{Y}}[\frac{1}{\pi}] \arrow[r] \arrow[d] & \mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}] \arrow[d, "\Delta"] \\
\mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}] \arrow[r, "(\mathrm{id},\varphi^*)"] & \mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}] \times \mathcal{V}_{\mathcal{Y}}[\frac{1}{\pi}].
\end{tikzcd}$$

We call objects of $\mathrm{Sht}_{\mathcal{Y}}[\frac{1}{\pi}](S)$ *isoshtukas* over S.

Definition 4.19. We let $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}} := (\mathrm{Bun}_{\mathrm{FF}})^{\mathrm{mer}}$ as an object in $\mathcal{S}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}_{1,E}^{\otimes,\mathrm{ex}})$. For $S \in \mathrm{Perf}^{\mathrm{aff}}$ we call $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)$ the stack of *meromorphic vector bundles on the relative Fargues–Fontaine curve* over S.

It follows from Remark 2.11 that we have a correspondence

$$\begin{tikzcd} \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}} \arrow[r, "\sigma"] \arrow[d, "\gamma"] & \mathrm{Bun}_{\mathrm{FF}} . \\ \mathfrak{B}^{\Diamond} & \end{tikzcd} \tag{4.1}$$

We give names to these maps.

Definition 4.20. (1) We call the map $\sigma : \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}} \to \mathrm{Bun}_{\mathrm{FF}}$ constructed in (4.1) the *special polygon map*.
(2) We call the map $\gamma : \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}} \to \mathfrak{B}^{\Diamond}$ constructed in (4.1) the *generic polygon map*.

We now study basic properties of $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}$. Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}^{\mathrm{aff}}$.

Proposition 4.21. *The following statements hold.*

(1) $\mathrm{Sht}_{\mathcal{Y}}[\frac{1}{\pi}]$ *is a v-separated presheaf in* $\mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}_{1,E}^{\otimes})$.
(2) The v-sheafification of $\mathrm{Sht}_{\mathcal{Y}}[\frac{1}{\pi}]$ *is equivalent to* $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}$ *in* $\mathcal{S}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}_{1,E}^{\otimes})$.
(3) A sequence in $\mathrm{Sht}_{\mathcal{Y}}[\frac{1}{\pi}]$ *is exact in* $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}$ *if and only if it is exact in* $\mathrm{Bun}_{\mathrm{FF}}^{(\mathrm{mer}_{\mathrm{pre}})}$.
(4) Exactness in $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}$ *can be verified on geometric points.*

Proof. The first two claims follow formally from Corollary 3.26 and Corollary 3.34.

Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}^{\mathrm{aff}}$. Fix $\Sigma := [\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3]$ a sequence with $\mathcal{E}_i \in \mathrm{Sht}_{\mathcal{Y}}[\frac{1}{\pi}](S)$. It is clear that if Σ is exact in $(\mathrm{Bun}_{\mathrm{FF}})^{(\mathrm{mer}_{\mathrm{pre}})}(S)$ then it is also exact in $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)$. Assume that Σ is exact in $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)$. By definition, this means that Σ is exact in $\mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(R'_{\mathrm{disc}}, R'^{+}_{\mathrm{disc}}))$ for a v-cover $S' := \mathrm{Spa}(R', R'^{+}) \to \mathrm{Spa}(R, R^+)$, and we need to show that Σ was already exact in $\mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(R_{\mathrm{disc}}, R^+_{\mathrm{disc}}))$.

Since $\mathrm{Sht}_{\mathcal{Y}}[\frac{1}{\pi}] \to \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}$ is fully-faithful, we deduce that the sequence is a complex. Let T' and T denote $\mathrm{Spd}(R'_{\mathrm{disc}}, R'^{+}_{\mathrm{disc}})$ and $\mathrm{Spd}(R_{\mathrm{disc}}, R^+_{\mathrm{disc}})$. We can verify exactness of Σ on geometric points of T. We warn the reader that although the map $S' \to S$ is a v-cover the map $T' \to T$ might no longer be surjective even at the level of topological spaces. Nevertheless, it is surjective on the loci where ϖ is topologically nilpotent for a pseudo-uniformizer $\varpi \in R^+$. Indeed these loci agree with S' and S, respectively. So it suffices to prove exactness of Σ on the complement of S in T.

Let $U = \mathrm{Spd}(R_{\mathrm{disc}}, R_{\mathrm{disc}})$, this is the locus in T where $|\varpi| \geq 1$. The complement of S in T, is the locus in which ϖ is not topologically nilpotent. If $x \in T \setminus S$ then there is a vertical generization y of x for which $|\varpi|_y = 1$. Indeed, x is represented by a geometric point $\mathrm{Spa}(C, C^+) \to \mathrm{Spd}(R_{\mathrm{disc}}, R^+_{\mathrm{disc}})$ and y is represented by the induced map $\mathrm{Spa}(C, O_C) \to \mathrm{Spd}(R_{\mathrm{disc}}, R^+_{\mathrm{disc}})$, so we see that ϖ maps to O_C. If ϖ lands in the maximal ideal of O_C then y (and consequently x) are in the locus in which ϖ is topologically nilpotent. Otherwise, the value of $|\varpi|_y = 1$. This shows that $T \setminus S \subseteq \overline{U}$ (where $\overline{U}$ denotes the closure), and since $S \subseteq T$ is an open subset, we must have $T \setminus S = \overline{U}$. Moreover, as we argued above $\overline{U} \setminus U$ consists of vertical specializations of elements in U, and the same holds for $\overline{U} \times \mathrm{Spd}\, E$ and $U \times \mathrm{Spd}\, E$. We can now conclude that Σ is exact over $\overline{U}$ if and only if it is exact over U. Indeed, for any affinoid perfectoid (A, A^+) the restriction functor

$$\mathcal{V}(Y_{\mathrm{Spa}(A,A^+)}) \to \mathcal{V}(Y_{\mathrm{Spa}(A,A^\circ)})$$

is an exact equivalence, so exactness can be verified on rank 1 points and in particular it is insensitive to passing to vertical generizations.

By hypothesis, Σ is exact when restricted to $\mathrm{Spd}(R', R')$. By Proposition 4.7, we may interpret Σ restricted to U as a sequence in $\mathfrak{B}(\mathrm{Spec}\, R)$ that becomes exact over $\mathfrak{B}(\mathrm{Spec}\, R')$. By Proposition 4.9, we can finish verifying exactness on closed points of $\mathrm{Spec}\, R$. But the map $\mathrm{Spec}\, R' \to \mathrm{Spec}\, R$ covers all closed points, since every maximal ideal of R supports a valuation that is continuous for the ϖ-adic topology. The kernel of any lift of such a valuation to R' maps to this maximal ideal.

For the final claim, we wish to prove that a sequence $\Sigma := [\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3]$ is exact in $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)$ if and only if for every geometric point $\overline{x} \to S$ the sequence $\Sigma_{\overline{x}}$ is exact. By definition, exactness can be verified v-locally. Hence, we may assume that $S = \mathrm{Spa}(R, R^+)$ is a product of points with $R^+ = \prod_{i\in I} C_i^+$ and that each $\mathcal{E}_j \in \mathrm{Sht}_{\mathcal{Y}}[\frac{1}{\pi}]$ for $j \in \{1, 2, 3\}$.

Since the map $R \to \prod_{i\in I} C_i$ is injective, we can deduce that Σ is a complex. We can argue as above to show that Σ is exact when interpreted as a sequence in $\mathrm{Bun}_{\mathrm{FF}}(\mathrm{Spd}(R_{\mathrm{disc}}, R^+_{\mathrm{disc}}))$. Namely, we show that Σ is exact on all points of $\mathrm{Spd}(R_{\mathrm{disc}}, R^+_{\mathrm{disc}})$. This is clear on the locus where ϖ is topologically nilpotent by our assumption. To verify exactness on $\mathrm{Spd}(R_{\mathrm{disc}}, R_{\mathrm{disc}})$ we interpret this as an object in $\mathfrak{B}(\mathrm{Spec}\, R)$ and we may check exactness on closed points. For any closed point, the map induced by the residue field $\mathrm{Spec}\, C \to \mathrm{Spec}\, R$ can be promoted to a geometric point $\mathrm{Spa}\, C \to \mathrm{Spa}(R, R^+)$ and the induced sequence in $\mathfrak{B}(\mathrm{Spec}\, C)$ is induced from the corresponding one in $\mathrm{Sht}_{\mathcal{Y}}[\frac{1}{\pi}](C, O_C)$, which is exact by assumption. $\square$

The following statement will be key for our purposes.

Corollary 4.22. *A sequence $\Sigma : [\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3]$ in $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)$ is exact if and only if its image in $\mathrm{Bun}_{\mathrm{FF}}(S)$ is exact.*

Proof. Since both statements are v-local and can be verified at the level of geometric points, we may assume $S = \mathrm{Spa}(C, C^+)$. We observe that if $T \subseteq S$ is the rank 1 point then $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(T) \simeq \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)$ and $\mathrm{Bun}_{\mathrm{FF}}(T) \simeq \mathrm{Bun}_{\mathrm{FF}}(S)$. Indeed, every moduli problem involved in the construction of these categories is insensitive to the ring of integral elements since $[\varpi]$ is inverted. We assume that $S = \mathrm{Spa}(C, O_C)$ and that $\mathcal{E}_i \in \mathrm{Sht}_{\mathcal{Y}}[\frac{1}{\pi}](S)$. Fix a pseudo-uniformizer $\varpi \in O_C$. In this case, exactness on $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)$ is equivalent to exactness of underlying vector bundles over $\mathrm{Spa}\, \mathbb{W}(O_C)_{\{\pi\cdot[\varpi]\neq 0\}}$, while exactness on $\mathrm{Bun}_{\mathrm{FF}}(S)$ is equivalent to exactness of underlying vector bundles over $Y_{(0,\infty),S} = \mathrm{Spa}\, \mathbb{A}_{\mathrm{inf}}(O_C)_{\{\pi\cdot[\varpi]\neq 0\}}$. We can analyze the behavior on the loci $\{|[\varpi]| \leq |\pi|\}$ and $\{|\pi| \leq |[\varpi]|\}$. In the former, the two spaces agree so their categories of vector bundles have the same exact structure. On the latter, we are comparing vector bundles over $\mathrm{Spec}\, B_{[0,1],S}[\frac{1}{\pi}]$ against vector bundles over $\mathcal{Y}_{(0,1],S}$.

Recall from [FS24, Theorem II.0.1, Corollary II.1.12] that $B_{[0,1],S}$ is a principal ideal domain, and that the closed ideals give rise to untilts of C. The claim now follows from the fact that the map of locally ringed topological spaces

$$f : \mathcal{Y}_{(0,1],S} \to \mathrm{Spec}\, B_{[0,1],S}[\frac{1}{\pi}]$$

covers every maximal ideal of the target and that $B_{[0,1],S}[\frac{1}{\pi}] \to \mathrm{H}^0(\mathcal{Y}_{(0,1],S}, \mathcal{O})$ is injective. This implies that f^* reflects exactness which is what we needed to show. $\square$

Proposition 4.23. *The following diagrams are Cartesian in $\mathcal{S}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,O_E})$ and $\mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes}_{1,E})$ respectively:*

$$\begin{tikzcd} \mathrm{Sht}_{\mathcal{Y}} \arrow[r] \arrow[d] & \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}} \arrow[d] & \mathrm{Sht}_{\mathcal{Y}}[\frac{1}{\pi}] \arrow[r] \arrow[d] & \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}} \arrow[d] \\ \mathrm{Sht}_{\mathbb{W}} \arrow[r] & \mathfrak{B}^{\diamondsuit} & \mathrm{Sht}_{\mathbb{W}}[\frac{1}{\pi}] \arrow[r] & \mathfrak{B}^{\diamondsuit} \end{tikzcd} \tag{4.2}$$

Here, the horizontal arrows in the right-hand square are the ones induced by sheafification.

Proof. The argument is a diagram chase whose key ingredients are Corollary 3.33 and Proposition 3.32. Since the two arguments are identical, we only provide the details for the first diagram. From Proposition 4.17 and Proposition 4.11 we have the following Cartesian diagrams

$$\begin{tikzcd} \mathrm{Sht}_{\mathcal{Y}} \arrow[r] \arrow[d] & (\mathcal{V}_Y)^{\mathrm{mer}} \arrow[d, "\Delta"] \\ \mathcal{V}_{\mathcal{Y}} \arrow[r] & (\mathcal{V}_Y)^{\mathrm{mer}} \times (\mathcal{V}_Y)^{\mathrm{mer}} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Sht}_{\mathbb{W}} \arrow[r] \arrow[d] & (\mathcal{V}_Y)^{\mathrm{An}} \arrow[d, "\Delta"] \\ \mathcal{V}_{\mathbb{W}} \arrow[r] & (\mathcal{V}_Y)^{\mathrm{An}} \times (\mathcal{V}_Y)^{\mathrm{An}}. \end{tikzcd} \tag{4.3}$$

Similarly, we obtain Cartesian diagrams

$$\begin{tikzcd} \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}} \arrow[r] \arrow[d] & (\mathcal{V}_Y)^{\mathrm{mer}} \arrow[d] \\ (\mathcal{V}_Y)^{\mathrm{mer}} \arrow[r] & (\mathcal{V}_Y)^{\mathrm{mer}} \times (\mathcal{V}_Y)^{\mathrm{mer}} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{B}^{\diamondsuit} \arrow[r] \arrow[d] & (\mathcal{V}_Y)^{\mathrm{An}} \arrow[d] \\ (\mathcal{V}_Y)^{\mathrm{An}} \arrow[r] & (\mathcal{V}_Y)^{\mathrm{An}} \times (\mathcal{V}_Y)^{\mathrm{An}}. \end{tikzcd} \tag{4.4}$$

Moreover, these four Cartesian diagrams can be organized in a commutative square of Cartesian diagrams. For any fixed $i \in \{left, right\}$ and $j \in \{upper, lower\}$, their (i, j)th corners form a commutative diagram, which we denote $C_{i,j}$. For example, $C_{left,upper}$ is the diagram that we wish to prove is Cartesian. Note that $C_{left,lower}$ is Cartesian by Corollary 3.33 and that for any $j \in \{upper, lower\}$ the square $C_{right,j}$ is automatically Cartesian, since the horizontal maps in it are isomorphisms. From this and the fact that taking limits commutes with each other, it formally follows that $C_{upper,left}$ is also Cartesian. □

5. Semi-stable filtrations

As we have justified in Proposition 4.21 (resp. Proposition 4.14), given $S \in \mathrm{Perf}^{\mathrm{aff}}$, the category $\mathrm{Sht}_{\mathcal{Y}}[\frac{1}{\pi}](S)$ (resp. $\mathrm{Sht}_{\mathbb{W}}[\frac{1}{\pi}](S)$) is a full subcategory of $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)$ (resp. $\mathfrak{B}^{\diamondsuit}(S)$) and it unambiguously inherits an exact structure. From this point on we will treat $\mathrm{Sht}_{\mathcal{Y}}[\frac{1}{\pi}]$ and $\mathrm{Sht}_{\mathbb{W}}[\frac{1}{\pi}]$ as objects in $\mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}_{1,E}^{\otimes,\mathrm{ex}})$. For fixed $S \in \mathrm{Perf}^{\mathrm{aff}}$, we may think of $\mathrm{Sht}_{\mathbb{W}}[\frac{1}{\pi}](S)$ as the full subcategory of those analytic isocrystals over S that admit a lattice. Similarly, we think of $\mathrm{Sht}_{\mathcal{Y}}[\frac{1}{\pi}](S)$ as the full subcategory of those meromorphic vector bundles over S that admit a lattice.

Definition 5.1. Fix $S \in \mathrm{Perf}^{\mathrm{aff}}$ with $S = \mathrm{Spa}(R, R^+)$. Given $\lambda \in \mathbb{Q}$ with $\lambda = \frac{m}{n}$ and $(m, n) = 1$, we let $\mathcal{O}(\lambda) \in \mathrm{Sht}_{\mathbb{W}}(S)[\frac{1}{\pi}]$ be given by the pair $(\mathbb{W}(R)[\frac{1}{\pi}]^n, M)$, where M is the matrix operator with $M \cdot e_i = e_{i+1}$ for $1 \le i \le n-1$ and $M \cdot e_n = \pi^{-m} e_1$. We call $\mathcal{O}(\lambda)$ the *simple standard analytic isocrystal of slope λ.*

We say that an object in $\mathcal{F} \in \mathfrak{B}^{\diamondsuit}(S)$ is *standard* if it is isomorphic to one of the form

$$\bigoplus_{\lambda \in \mathbb{Q}} \mathcal{O}(\lambda)^{m_\lambda},$$

where $m : \mathbb{Q} \to \mathbb{N}$ is a multiplicity function with finite support.

Remark 5.2. We warn the reader that our parametrization of standard analytic isocrystals reverses the signs of the parametrization of "usual" isocrystals used on most classical conventions.

For us a *Newton polygon* is a function $f : \mathbb{Q} \to \mathbb{Z}_{\ge 0}$ with $f^{-1}(\mathbb{Z}_{>0})$ finite. Its slopes are the values $x \in \mathbb{Q}$ with $f(x) \neq 0$ and the multiplicity of the slope x is $f(x)$. We denote by $\mathcal{N}$ the set of all Newton polygons. Then $\mathcal{N}$ is endowed with the partial order $f \le g$ if and only if

$$\sum_{x \in \mathbb{Q}} f(x)x = \sum_{x \in \mathbb{Q}} g(x)x$$

and for all $x \in \mathbb{Q}$ one has

$$\sum_{y \geq x} f(y)y \leq \sum_{y \geq x} g(y)y\,.$$

We say a Newton polygon is *semi-stable* if it has a single slope. We let $\mathcal{N}^{ss} \subseteq \mathcal{N}$ denote the subset of semi-stable polygons, these are the minimal elements in $\mathcal{N}$.

We wish to use $\mathcal{N}$ to stratify $\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}$. Before we do this, we make the following sanity check. It says that over geometric points analytic isocrystals and meromorphic vector bundles admit a lattice. Alternatively, it says that sheafification does not change the naive value on geometric points.

Lemma 5.3. *Let $S = \mathrm{Spa}(C, C^+)$ be a geometric point. We have the following equivalences in $\mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E}$:*

(1) $\mathfrak{B}^{\Diamond}(S) \simeq \mathrm{Sht}_{\mathbb{W}}[\frac{1}{\pi}](S) \simeq \mathfrak{B}(\mathrm{Spec}\,\overline{\mathbb{F}}_q)$.

(2) $\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}(S) \simeq \mathrm{Sht}_{\mathcal{Y}}[\frac{1}{\pi}](S)$.

Proof. The second claim follows from the first one and Proposition 4.23. By inspection, $\mathrm{Sht}_{\mathbb{W}}[\frac{1}{\pi}](S) \simeq \mathfrak{B}(\mathrm{Spec}\,\bar{\mathbb{F}}_q)$ and these categories sit fully-faithfully inside $\mathfrak{B}^{\Diamond}(S)$. It suffices to show that $\mathrm{Sht}_{\mathbb{W}}[\frac{1}{\pi}](S) \subseteq \mathfrak{B}^{\Diamond}(S)$ is also essentially surjective. Fix $\mathcal{E} \in \mathfrak{B}^{\Diamond}(S)$. Then there is a v-cover $f : S' \to S$ such that $f^*\mathcal{E} \in \mathrm{Sht}_{\mathbb{W}}[\frac{1}{\pi}](S')$. We may assume that $S' = \mathrm{Spa}(C', C'^+)$ is a geometric point. In this case, $f^*\mathcal{E}$ is of the form $\bigoplus_{\lambda\in\mathbb{Q}} \mathcal{O}(\lambda)^{m_\lambda}$. Moreover, $\mathcal{E} \in \mathrm{Desc.}(\mathrm{Sht}_{\mathbb{W}}[\frac{1}{\pi}], S'/S)$. The descent datum can be recorded by an automorphism of $\bigoplus_{\lambda\in\mathbb{Q}} \mathcal{O}(\lambda)^{m_\lambda}$ over $\mathrm{Spec}\,(C'\hat{\otimes}_C C')$. Now, this is a connected affine scheme by [Sch17, Lemma 14.6]. The descent datum is necessarily given by a constant function in $\prod_{\lambda\in\mathbb{Q}}\mathrm{Aut}(\mathcal{O}(\lambda)^{m_\lambda})$ and since it has to satisfy the cocycle condition upon pullback to $\mathrm{Spec}\,(C'\hat{\otimes}_C C'\hat{\otimes}_C C')$, this function is necessarily the identity. Consequently, $\mathcal{E}$ is isomorphic to $\bigoplus_{\lambda\in\mathbb{Q}}\mathcal{O}(\lambda)^{m_\lambda}$ already over S. □

If S is a geometric point, then isomorphism classes of objects in $\mathrm{Bun}_{\mathrm{FF}}(S)$ and $\mathfrak{B}^{\Diamond}(S)$ are both in natural bijection with $\mathcal{N}$. Indeed, for analytic isocrystals this is Lemma 5.3, and for vector bundles on the Fargues–Fontaine curve this is proven in [Far20] and [Ans19, Theorem 3.11]. In other words, we have canonical bijections

$$\nu : \mathfrak{B}^{\Diamond}(S) \xrightarrow{\simeq} \mathcal{N} \xleftarrow{\simeq} \mathrm{Bun}_{\mathrm{FF}}(S) : \nu\,.$$

Definition 5.4. Given $S \in \mathrm{Perf}^{\mathrm{aff}}$ and $\mathcal{E} \in \mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}(S)$ we define two functions $\gamma_{\mathcal{E}}, \sigma_{\mathcal{E}} : |S| \to \mathcal{N}$ which we call the *generic polygon* and *special polygon*, respectively. For $x \in |S|$ we choose a geometric point $\overline{x} \to S$ over x and we let $\gamma_{\mathcal{E}}(x) := \nu(\gamma(\mathcal{E}_{\overline{x}}))$. We define $\sigma_{\mathcal{E}}$ similarly.

Remark 5.5. Using a different language, Kedlaya proves that for any $\mathcal{E} \in \mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}$ we have $\gamma_{\mathcal{E}} \geq \sigma_{\mathcal{E}}$, see [Ked05, Prop. 5.5.1]. This a key step in Kedlaya–Liu's proof of the semicontinuity theorem [KL13, Theorem 7.4.5].

Definition 5.6. Let $\mathcal{E} \in \mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}(S)$ with constant rank and image $\mathcal{F} \in \mathfrak{B}^{\Diamond}(S)$ under the map $\gamma : \mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}(S) \to \mathfrak{B}^{\Diamond}(S)$.

(1) We say that $\mathcal{F}$ is *locally standard* if its Newton polygon is locally constant.
(2) We say $\mathcal{E}$ is *generically locally standard* if $\mathcal{F}$ is locally standard, equivalently if $\gamma_{\mathcal{E}}$ is locally constant.

We let $(\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}})^{\mathrm{loc}}(S)$ and $(\mathfrak{B}^{\Diamond})^{\mathrm{loc}}(S)$ denote the full subcategories described above.

Remark 5.7. The functors

$$(\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}})^{\mathrm{loc}},\ (\mathfrak{B}^{\Diamond})^{\mathrm{loc}} \in \mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E})$$

are still v-sheaves since the condition defining them can be verified v-locally. Indeed, for a v-cover $f : \mathrm{Spa}(R_1, R_1^+) \to \mathrm{Spa}(R_2, R_2^+)$ and an open and closed decomposition $\mathrm{Spa}(R_1, R_1^+) = \coprod_{\gamma\in\mathcal{N}} U_\gamma$ we must have $U_\gamma = f^{-1}(f(U_\gamma))$ since the generic Newton polygon is an invariant of the geometric points of $\mathrm{Spa}(R_2, R_2^+)$. Since $|f|$ is a quotient map, $f(U_\gamma)$ is also closed and open in $|\mathrm{Spa}(R_2, R_2^+)|$ so the Newton polygon on $\mathrm{Spa}(R_2, R_2^+)$ is locally constant.

Definition 5.8. Let $S = \mathrm{Spa}(R, R^+)$. We say that an object $(\mathcal{F}, \Phi) \in \mathrm{Sht}_{\mathbb{W}}(S)$ is *anti-effective* if the isomorphism $\Phi^{-1} : \mathcal{F} \to \varphi^*\mathcal{F}$ extends to a map $\Psi : \mathcal{F} \to \varphi^*\mathcal{F}$ defined over $\mathrm{Spec}\, \mathbb{W}(R)$. An object in $\mathcal{E} \in \mathrm{Sht}_{\mathcal{Y}}(S)$ is *anti-effective* if its image in $\mathrm{Sht}_{\mathbb{W}}(S)$ is anti-effective.

Proposition 5.9. *Let $\mathcal{E} \in \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)$ such that the function $\gamma_{\mathcal{E}}$ is constant and such that its smallest slope is* 0*. Then it lifts v-locally to an anti-effective crystalline shtuka.*

Proof. By Proposition 4.23 it suffices to prove that locally standard analytic isocrystals of smallest slope 0 lift v-locally to an anti-effective Dieudonné module. Working v-locally, we may assume $\gamma(\mathcal{E}) \in \mathrm{Sht}_{\mathbb{W}}[\frac{1}{\pi}](S)$, and since $\gamma(\mathcal{E})$ is locally standard, we may by [HK22, Theorem 2.11] even assume $\gamma(\mathcal{E}) \cong \oplus_{i=1}^n \mathcal{O}(\lambda_i)^{m_i}$. By assumption, $\lambda_i \geq 0$ for i. The standard models of $\mathcal{O}(\lambda_i)$ already define an anti-effective crystalline shtuka by inspection of Definition 5.1. □

Lemma 5.10. *Suppose that $S = \mathrm{Spa}(R, R^+)$ is a product of points. Let $(\mathcal{E}, \Phi) \in \mathrm{Sht}_{\mathcal{Y}}(S)$ be anti-effective, then*

$$\mathrm{Hom}_{\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}}(\mathcal{O}, \mathcal{E}) = \mathrm{Hom}_{\mathfrak{B}^\diamondsuit}(\mathcal{O}, \gamma(\mathcal{E}))\,.$$

Moreover, if $f \in \mathrm{Hom}_{\mathfrak{B}^\diamondsuit}(\mathcal{O}, \gamma(\mathcal{E}))$ defines a sub-isocrystal $\mathcal{O} \subseteq \mathcal{E}$, then the corresponding lift also defines a sub-bundle $\mathcal{O} \subseteq \mathcal{E}$ in $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}$.

Proof. By Proposition 4.21, we may compute $\mathrm{Hom}_{\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}}(\mathcal{O}, \mathcal{E})$ in $\mathrm{Sht}_{\mathcal{Y}}[\frac{1}{\pi}]$. Since $B_{[0,r],S} \subseteq \mathbb{W}R$, the map

$$\mathrm{Hom}_{\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}}(\mathcal{O}, \mathcal{E}) \to \mathrm{Hom}_{\mathfrak{B}^\diamondsuit}(\mathcal{O}, \gamma(\mathcal{E}))$$

is injective. To prove surjectivity, we fix a basis of $\beta : \mathcal{O}^n \to \mathcal{E}$ over $\mathcal{Y}_{[0,\frac{q}{N}]}$ for some $N \in \mathbb{N}$. This induces a basis $\varphi^*\beta : \mathcal{O}^n \to \varphi^*\mathcal{E}$ over $\mathcal{Y}_{[0,\frac{1}{N}]}$, let $r = \frac{1}{N}$. Since $(\mathcal{E}, \Phi)$ is anti-effective, we can think of $(\mathcal{E}, \Phi)$ through β and $\varphi^*\beta$ as a matrix $M \in \mathrm{GL}_n(B^R_{[0,r]})$ such that

$$M^{-1} \in \mathrm{GL}_n(B^R_{[0,r]}[\frac{1}{\pi}]) \cap M_{n\times n}(\mathbb{W}R)\,.$$

A map $f \in \mathrm{Hom}_{\mathfrak{B}^\diamondsuit}(\mathcal{O}, \gamma(\mathcal{E}))$ can then be thought of as a vector $v \in \mathbb{W}(R)[\frac{1}{\pi}]^n$ satisfying the equation

$$M\varphi v = v\,.$$

On the other hand, $v \in \mathrm{Hom}_{\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}}(\mathcal{O}, \mathcal{E})$ if and only if $v \in B_{[0,s]}[\frac{1}{\pi}]$ for some $s > 0$. Indeed, we can use φ-equivariance to extend this map along $\mathcal{Y}_{(\frac{s}{2},\infty)}$. Replacing v by $\pi^N \cdot v$, we may assume $v \in \mathbb{W}(R)^n$.

We fix a norm of $|\cdot| : R \to \mathbb{R}$ inducing the topology of R with $|\varpi| = \frac{1}{q}$ and define a function $|\cdot|_k : \mathbb{W}R \to \mathbb{R}$ by the formula

$$\sum_{i=0}^{\infty} [a_i]\pi^i \mapsto \sup_{0\leq i\leq k} |a_i|.$$

This definition extends to $M_{n\times n}(\mathbb{W}R)$ and $(\mathbb{W}R)^n$ by taking supremum over the entries. By the strong triangle inequality, and because $M^{-1} \in M_{n\times n}(\mathbb{W}R)$, we see that for every $k \in \mathbb{N}$ the inequality $|M^{-1} \cdot v|_k \leq |M^{-1}|_k \cdot |v|_k$ holds and by inspection $|\varphi v|_k = |v|_k^q$. From this we deduce that $|v|_k^{q-1} \leq |M^{-1}|_k$. Let $m_{ij} \in B^R_{[0,r]}$ denote the (i,j) entry of M^{-1} and write $m_{ij} = \sum_{l=0}^{\infty} [m_{ijl}]\pi^l$. The sequences m_{ijl} all satisfy that $\lim_{l\mapsto\infty} |m_{ijl}| \cdot (\frac{1}{q})^{N\cdot l} = 0$. Now, Lemma 5.11 shows that $\lim_{l\mapsto\infty} |M^{-1}|_l \cdot (\frac{1}{q})^{N\cdot l} = 0$ and in particular that $\lim_{l\mapsto\infty} |v|_l \cdot (\frac{1}{q})^{N\cdot(q-1)\cdot l} = 0$, which implies that $v \in (B^R_{[0,\frac{1}{N\cdot(q-1)}]})^n$ as we needed to show.

By Proposition 4.21, the last claim can be verified at the level of geometric points. Consider the ideal I in $B^C_{[0,1]}$ generated by the entries of v. Since $B^C_{[0,1]}$ is a principal ideal domain, the zero locus of I consists of finitely many closed points in Spec $B^C_{[0,1]}$. Moreover, the zero locus is φ-equivariant so it is at worst the ideal cut out by π, but then it avoids Spec $B^C_{[0,1]}[\frac{1}{\pi}]$. □

Lemma 5.11. *Let I be a finite set and ρ a number with $0 < \rho < 1$. For each $i \in I$, let $(b_{i,j})_{j\geq 0}$ be a sequence in $\mathbb{R}_{\geq 0}$ such that $\lim_{j\mapsto\infty} b_{i,j}\cdot\rho^j = 0$. For each $j \geq 0$, let $B_j = \max_{i\in I,j'\leq j}\{b_{i,j'}\}$. Then $\lim_{j\mapsto\infty} B_j\cdot\rho^j = 0$.*

Proof. This easily reduces to the case $I = \{1\}$. Fix $\varepsilon > 0$. By assumption, there is some $j_{\varepsilon,0} > 0$ such that for all $j \geq j_{\varepsilon,0}$, we have $b_j\rho^j < \varepsilon$. Put

$$\lambda = \max_{j'<j_{\varepsilon,0}} b_{j'}\rho^{j'}\,.$$

We now pick a big enough j_ε, such that $\rho^{j_\varepsilon - j_{\varepsilon,0}}\lambda < \varepsilon$. Then for any $j \geq j_\varepsilon$ we have

$$\begin{aligned} B_j\rho^j &= \max_{j'\leq j}\{b_{j'}\rho^j\} \\ &= \max\{\max_{j'<j_{\varepsilon,0}}\{b_{j'}\rho^{j'}\rho^{j-j'}\}, \max_{j_{\varepsilon,0}\leq j'\leq j}\{b_{j'}\rho^{j'}\rho^{j-j'}\}\} < \varepsilon\,. \end{aligned}$$

Indeed, if $j' < j_{\varepsilon,0}$, then $b_{j'}\rho^{j'}\rho^{j-j'} \leq \lambda\rho^{j-j'} \leq \lambda\rho^{j_\varepsilon - j_{\varepsilon,0}} < \varepsilon$ (as $\rho < 1$ and $j - j' \geq j_\varepsilon - j_{\varepsilon,0}$); and if $j' > j_{\varepsilon,0}$, then $b_{j'}\rho^{j'} < \varepsilon$ and $\rho^{j-j'} < 1$. □

Definition 5.12. Let $S \in \mathrm{Perf}^{\mathrm{aff}}$. For a fixed $\lambda \in \mathbb{Q}$, we let $(\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}})_\lambda(S) \subseteq (\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}})^{\mathrm{loc}}(S)$, (resp. $(\mathfrak{B}^\diamondsuit)_\lambda(S) \subseteq (\mathfrak{B}^\diamondsuit)^{\mathrm{loc}}(S)$, resp. $\mathrm{Bun}_{\mathrm{FF}}(S)_\lambda \subseteq \mathrm{Bun}_{\mathrm{FF}}$) denote the full subcategories of objects whose generic Newton polygon function $\gamma_{(-)}$ (resp. Newton polygon function ν) attaches to each geometric point of S a constant polygon of slope λ. We call objects in these subcategories semi-stable of slope λ.

Proposition 5.13. *For all $\lambda \in \mathbb{Q}$ the maps $(\mathfrak{B}^\diamondsuit)_\lambda \xleftarrow{\gamma} (\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}})_\lambda \xrightarrow{\sigma} (\mathrm{Bun}_{\mathrm{FF}})_\lambda$ are exact equivalences of sheaves of E-linear exact categories.*

Remark 5.14. We note that the categories $(\mathfrak{B}^\diamondsuit)_\lambda$, $(\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}})_\lambda$, $(\mathrm{Bun}_{\mathrm{FF}})_\lambda$ are not stable under $\otimes$-products.

Proof. To prove that γ and σ are equivalences, it suffices to show that they are fully-faithful. Indeed, by [CS17, Proposition 4.3.13] (resp. [FS24, Theorem I.3.4]) every object of $(\mathfrak{B}^\diamondsuit)_\lambda$ (resp. $(\mathrm{Bun}_{\mathrm{FF}})_\lambda$) is pro-étale locally isomorphic to $\mathcal{O}(\lambda)^m$ which is already in $\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}(\mathrm{Spd}\,\overline{\mathbb{F}}_q)$. Then, Lemma A.2 allows us to conclude. To show full-faithfulness, we may pass to internal $\underline{\mathrm{Hom}}$-objects and take global sections which reduces us to prove that the maps

$$\mathrm{Hom}_{\mathfrak{B}^\diamondsuit}(\mathcal{O},\gamma\mathcal{E}) \xleftarrow{\gamma} \mathrm{Hom}_{\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}}(\mathcal{O},\mathcal{E}) \xrightarrow{\sigma} \mathrm{Hom}_{\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}}(\mathcal{O},\sigma\mathcal{E})$$

are isomorphisms for all $\mathcal{E} \in \mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}(S)_0$.

Let us show γ is fully-faithful. We may instead prove fully-faithfulness of the maps $\mathrm{Sht}_{\mathcal{Y}}[\frac{1}{\pi}](S) \to \mathrm{Sht}_{\mathbb{W}}[\frac{1}{\pi}](S)$ when restricted to the slope 0-locus, since this will pass to the sheafification. More precisely, given $(\mathcal{E},\Phi) \in \mathrm{Sht}_{\mathcal{Y}}[\frac{1}{\pi}](S)$ with image in $(\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}})_0 \subset \mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}$, we have a bijection

$$\mathrm{Hom}_{\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}}(\mathcal{O},\mathcal{E}) \to \mathrm{Hom}_{\mathfrak{B}^\diamondsuit}(\mathcal{O},\gamma\mathcal{E})\,.$$

Indeed, this follows from Proposition 5.9 and Lemma 5.10.

Let $\underline{E} - \mathrm{Loc}$ denote the category of pro-étale $\underline{E}$-local systems. By [CS17, Proposition 4.3.13], [FS24, Theorem I.3.4] and the argument above we get to a commutative diagram in $\mathrm{Cat}^{\otimes}_{1,E}$ of the form

$$\begin{tikzcd}
\underline{E}-\mathrm{Loc}(S) \arrow[dr] \arrow[drr, bend left, "\simeq"] \arrow[ddr, bend right, "\simeq"'] & & \\
& \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)_0 \arrow[r] \arrow[d, "\simeq"] & \mathrm{Bun}_{\mathrm{FF}}(S)_0 . \\
& \mathfrak{B}^{\diamondsuit}(S)_0 &
\end{tikzcd}$$

This is enough to conclude that the remaining two arrows are also equivalences.

By Lemma 3.1, Proposition 4.15 and Corollary 4.22, exactness of the equivalences can be checked on geometric points. By Lemma 5.3, the three categories are equivalent to a semi-simple category over a geometric point, so the exact structure is the one inherited from the additive structure and any equivalence preserves it. □

We consider $\mathbb{Q}$-filtered meromorphic vector bundles (resp. vector bundles, resp. analytic isocrystals). That is, we consider sequences of the form $\{\mathcal{E}_r\}_{r\in\mathbb{Q}} \in \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)$ (resp. $\{\mathcal{E}_r\}_{r\in\mathbb{Q}} \in \mathrm{Bun}_{\mathrm{FF}}(S)$, resp. $\{\mathcal{E}_r\}_{r\in\mathbb{Q}} \in \mathfrak{B}^{\diamondsuit}(S)$) with $\mathcal{E}_r \subseteq \mathcal{E}_s$ when $r < s$ such that $\mathcal{E}_r/\mathcal{E}_{<r} = 0$ for all but finitely many $r \in \mathbb{Q}$ and such that

$$0 \to \mathcal{E}_r \to \mathcal{E}_s \to \mathcal{E}_r/\mathcal{E}_s \to 0$$

is an exact sequence in $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)$ (resp. $\mathrm{Bun}_{\mathrm{FF}}(S)$, resp. $\mathfrak{B}^{\diamondsuit}(S)$). By hypothesis, there is $N \gg 0$ such that $\mathcal{E}_s = \mathcal{E}_N$ for every $s > N$ and we call $\mathcal{E}_N$ *the underlying vector bundle* of $\{\mathcal{E}_r\}_{r\in\mathbb{Q}}$.

Definition 5.15. We say that a $\mathbb{Q}$-filtered meromorphic vector bundle (resp. a vector bundle, resp. analytic isocrystal) is a *semi-stable filtration* if $\mathcal{E}_r/\mathcal{E}_{<r}$ is semi-stable of slope r in the sense of Definition 5.12. We let $\mathcal{F}il_{\mathrm{ss}}^{\mathrm{mer}}(S)$ (resp. $\mathcal{F}il_{\mathrm{ss}}^{\sigma}(S)$, resp. $\mathcal{F}il_{\mathrm{ss}}^{\gamma}(S)$) denote the categories whose objects are semi-stable filtrations and whose morphisms are maps in $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)$ (resp. $\mathrm{Bun}_{\mathrm{FF}}(S)$, resp. $\mathfrak{B}^{\diamondsuit}(S)$) that respect the filtration. We endow these categories with the exact structure inherited from their underlying vector bundle. Moreover, the tensor product of underlying vector bundles inherits a filtration with

$$(\mathcal{E} \otimes \mathcal{F})_r = \sum_{r_{\mathcal{E}}+r_{\mathcal{F}}=r} \mathcal{E}_{r_{\mathcal{E}}} \otimes \mathcal{F}_{r_{\mathcal{F}}}$$

such that

$$(\mathcal{E} \otimes \mathcal{F})_r/(\mathcal{E} \otimes \mathcal{F})_{<r} = \bigoplus_{r_{\mathcal{E}}+r_{\mathcal{F}}=r} \mathcal{E}_{r_{\mathcal{E}}}/\mathcal{E}_{<r_{\mathcal{E}}} \otimes \mathcal{F}_{r_{\mathcal{F}}}/\mathcal{F}_{<r_{\mathcal{F}}} .$$

Proposition 5.16. *The natural map $\mathcal{F}il_{\mathrm{ss}}^{\mathrm{mer}} \to \mathcal{F}il_{\mathrm{ss}}^{\sigma}$ is a $\otimes$-exact equivalence of v-stacks.*

Proof. Full-faithfulness: Let $\{\mathcal{E}_r\}_{r\in\mathbb{Q}}$ and $\{\mathcal{F}_r\}_{r\in\mathbb{Q}}$ be in $\mathcal{F}il_{\mathrm{ss}}^{\mathrm{mer}}(S)$, with underlying meromorphic vector bundles $\mathcal{E}$ and $\mathcal{F}$. The internal $\underline{\mathrm{Hom}}$-bundle

$$\mathcal{H} := \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{F}) = \mathcal{E}^{\vee} \otimes \mathcal{F}$$

is naturally endowed with a $\mathbb{Q}$-filtration $\{\mathcal{H}_r\}_{r\in\mathbb{Q}}$. It is not hard to verify that $\{\mathcal{H}_r\}_{r\in\mathbb{Q}}$ is a semi-stable filtration. Moreover, we have an identification

$$\mathrm{Hom}_{\mathcal{F}il_{\mathrm{ss}}^{\mathrm{mer}}}(\{\mathcal{E}_r\}_{r\in\mathbb{Q}}, \{\mathcal{F}_r\}_{r\in\mathbb{Q}}) = \mathrm{Hom}_{\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}}(\mathcal{O}, \mathcal{H}_{\leq 0}) .$$

Analogously,

$$\mathrm{Hom}_{\mathcal{F}il_{\mathrm{ss}}}(\{\mathcal{E}_r\}_{r\in\mathbb{Q}}, \{\mathcal{F}_r\}_{r\in\mathbb{Q}}) = \mathrm{Hom}_{\mathrm{Bun}_{\mathrm{FF}}}(\mathcal{O}, \mathcal{H}_{\leq 0}) .$$

Since $\{\mathcal{H}_r\}_{r\in\mathbb{Q}}$ is semistable, one can prove inductively on the support of $\{\mathcal{H}_r\}_{r\in\mathbb{Q}}$ that

$$\mathrm{Hom}_{\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}}(\mathcal{O}, \mathcal{H}_{\leq r}) = 0 = \mathrm{Hom}_{\mathrm{Bun}_{\mathrm{FF}}}(\mathcal{O}, \mathcal{H}_{\leq r})$$

for all $r < 0$. To prove fully-faithfulness it suffices to show that

$$\mathrm{Hom}_{\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}}(\mathcal{O}, \mathcal{H}_{\leq 0}/\mathcal{H}_{<0}) \cong \mathrm{Hom}_{\mathrm{Bun}_{\mathrm{FF}}}(\mathcal{O}, \mathcal{H}_{\leq 0}/\mathcal{H}_{<0}).$$

But since $\mathcal{H}_{\leq 0}/\mathcal{H}_{<0}$ is semi-stable of slope 0, the result follows directly from Proposition 5.13.

Essential surjectivity: Let $\{\mathcal{E}_r\} \in \mathcal{F}il^{\sigma}_{\mathrm{ss}}$ with underlying vector bundle $\mathcal{E}$ of rank n. If E_s is the degree s unramified extension of E, then objects in $\mathrm{Bun}_{\mathrm{FF}}$ can be constructed by descent from objects in $\mathrm{Bun}_{\mathrm{FF},E_s}$, and by fully-faithfulness a descent datum in $\mathcal{F}il^{\sigma}_{\mathrm{ss}}$ agrees with a descent datum in $\mathcal{F}il^{\mathrm{mer}}_{\mathrm{ss}}$. This reduces us to show that for a suitable s, the pullback of the filtered vector bundle $\{\mathcal{E}_r\}$ to the Fargues–Fontaine curve associated with E_s lies in the essential image of $\mathcal{F}il^{\mathrm{mer}}_{\mathrm{ss}} \to \mathcal{F}il^{\sigma}_{\mathrm{ss}}$. This pullback has the effect of multiplication by s on the slopes. Picking s to be the product of all denominators appearing on the slopes of $\{\mathcal{E}_r\}$, we may thus assume that the support of the filtration is contained in $\mathbb{Z}$. Since essential surjectivity can now be proved v-locally by Lemma A.2, we may think of every bundle $\mathcal{E}_r$ as a free module M_r over $B^R_{[1,q]}$ with φ-glueing data over $B^R_{[1,1]}$. We may even assume that the graded pieces $\mathcal{E}_N/\mathcal{E}_{<N}$ are isomorphic to $\mathcal{O}(N)^{m_N}$. We may choose bases for the M_r over $B^R_{[1,q]}$ compatible with the filtration and in such a way that after transferring the Frobenius structure to $\mathcal{O}^n$, the induced N-graded pieces are given by diagonal matrices of the form π^{-N}. Thus $\{\mathcal{E}_r\}$ is represented by an upper block-diagonal matrix $A \in M_{n\times n}(B^R_{[1,1]})$, with diagonal blocks of the form $\pi^{-N} \cdot \mathrm{Id}_{m_N,m_N}$, describing the Frobenius structure. Let $P \subseteq \mathrm{GL}_n$ denote the parabolic subgroup (containing the upper triangular matrices) corresponding to the block-diagonal shape of A.

Now it suffices to prove the claim that there is a matrix $A_\infty \in P(B^R_{[0,1]}[\frac{1}{\pi}])$ and a matrix $U \in P(B^R_{[1,q]})$ with $U^{-1}A_\infty\varphi(U) = A$. Indeed, such U defines a (not necessarily meromorphic) isomorphism of the filtered vector bundle $\{\mathcal{E}_r\}$ with the filtered vector bundle represented by A_∞; now, π is only meromorphically inverted in $B^R_{[0,1]}[\frac{1}{\pi}]$, hence A_∞ in fact defines a meromorphic filtered vector bundle. Now our claim follows from Lemma 5.20 below. □

Before proving the remaining Lemma 5.20, we need some preparations.

Lemma 5.17. *We have* $B^R_{[1,1]} = B^R_{[0,1]}[\frac{1}{\pi}] + [\varpi]B^R_{[1,\infty]}$.

Proof. Let $A_1 = \mathbb{W}(R^+)[\frac{\pi}{[\varpi]}]$, $A_2 = \mathbb{W}(R^+)[\frac{[\varpi]}{\pi}]$ and $A_{12} = \mathbb{W}(R^+)[\frac{\pi}{[\varpi]}, \frac{[\varpi]}{\pi}]$. We have $B^R_{[1,1]} = (A_{12})^\wedge_\pi[\frac{1}{\pi}]$, $B^R_{[0,1]} = (A_1)^\wedge_{[\varpi]}[\frac{1}{[\varpi]}]$ and $B^R_{[1,\infty]} = (A_2)^\wedge_\pi[\frac{1}{\pi}]$. After multiplication with a big enough power of π, it suffices to show that any element of $(A_{12})^\wedge_\pi$ can be written as a sum of an element of $(A_1)^\wedge_{[\varpi]}$ and an element of $\frac{[\varpi]}{\pi} \cdot (A_2)^\wedge_\pi$.

For any $n \geq 1$, let $I_n = \{(i,j) \in \mathbb{Z}^2 : 0 \leq i < n\}$ and let

$$S_n \subseteq \prod_{(i,j)\in I_n} R^+$$

be the subset of all sequences $a = (a_{ij})_{ij}$ for which $a_{ij} = 0$ except for finitely many $(i,j) \in I_n$. Let also $S_n^+ \subseteq S_n$ (resp. $S_n^- \subseteq S_n$) be the subset of all sequences for which $a_{ij} = 0$ unless $j \geq 0$ (resp. $a_{ij} = 0$ unless $j < 0$). There is a commutative diagram D_n of sets

$$\begin{tikzcd} S_n^+ \arrow[r] \arrow[d] & S_n \arrow[d] & S_n^- \arrow[l] \arrow[d] \\ A_1/[\varpi]^n A_1 \arrow[r] & A_{12}/\pi^n A_{12} & \frac{[\varpi]}{\pi} \cdot (A_2/\pi^n A_2) \arrow[l] \end{tikzcd}$$

(note that $A_{12}/\pi^n A_{12} = A_{12}/[\varpi]^n A_{12}$), where the upper horizontal maps are the defining inclusions, the lower horizontal maps are induced by the natural ring maps $A_1 \to A_{12} \leftarrow A_2$ (and the inclusion of the ideal $\frac{[\varpi]}{\pi} A_2 \subseteq A_2$) and the vertical maps are given by sending $(a_{ij})_{ij}$ to $\sum_{ij} [a_{ij}]\pi^i \cdot (\frac{\pi}{[\varpi]})^j$.

We make three observations, which immediately follow from the explicit definition of the vertical maps: first, the middle vertical map is surjective. Second, there is an obvious map $D_{n+1} \to D_n$ of commutative diagrams and the resulting diagram is commutative. Third, when we define the map $+ : S_n^+ \times S_n^- \to S_n$ by $(a+b)_{ij} = a_{ij}$ if $j \geq 0$ and $(a+b)_{ij} = b_{ij}$ if $j < 0$, then the resulting diagram

$$\begin{tikzcd} S_n^+ \times S_n^- \arrow[r, "+"] \arrow[d] & S_n \arrow[d] \\ A_1/[\varpi]^n A_1 \times \frac{[\varpi]}{\pi} \cdot (A_2/\pi^n A_2) \arrow[r, "+"] & A_{12}/\pi^n A_{12} \end{tikzcd}$$

is commutative.

Let now $S = \lim_n S_n$ and $S^\pm = \lim_n S_n^\pm$. Explicitly, $S \subseteq \prod_{(i,j) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}} R^+$ is the subset of all sequences $(a_{ij})_{ij}$ satisfying the following condition: for each i there is some $j(i) \geq 0$ such that $a_{ij} = 0$ unless $|j| < j(i)$ and S^+ and S^- are corresponding subsets of S. Passing to the limit over all $n > 0$, we obtain a commutative diagram

$$\begin{tikzcd} S^+ \arrow[r] \arrow[d] & S \arrow[d] & S^- \arrow[l] \arrow[d] \\ (A_1)^\wedge_{[\varpi]} \arrow[r] & (A_{12})^\wedge_\pi & \frac{[\varpi]}{\pi} \cdot (A_2)^\wedge_\pi . \arrow[l] \end{tikzcd}$$

Moreover, we also get the commutative diagram

$$\begin{tikzcd} S^+ \times S^- \arrow[r, "+"] \arrow[d] & S \arrow[d] \\ (A_1)^\wedge_{[\varpi]} \times \frac{[\varpi]}{\pi} \cdot (A_2)^\wedge_\pi \arrow[r, "+"] & (A_{12})^\wedge_\pi , \end{tikzcd}$$

where the lower horizontal map is the restriction of the addition map $B_{[0,1]} \times \frac{[\varpi]}{\pi} \cdot B_{[1,\infty]} \to B_{[1,1]}$ and the upper horizontal map is defined in the same way as $S_n^+ \times S_n^- \to S_n$. Now, one can concretely verify that $S^+ \times S^- \to S$ and $S \to (A_{12})^\wedge_\pi$ are surjective using Lemma 5.18. This implies that the lower horizontal map in the diagram is surjective as well, which is precisely what we had to show. □

Lemma 5.18. *Let* $A_{(-)}, B_{(-)} : \mathbb{N}^{\mathrm{op}} \to \mathrm{Sets}$ *denote two functors and denote by* $g_n^A : A_n \to A_{n-1}$ *and* $g_n^B : B_n \to B_{n-1}$ *the image of the unique morphism* $(n-1) \to (n)$ *when we treat the partial order* $\mathbb{N}$ *as a category. Let* $f : A \to B$ *be a natural transformation that is pointwise surjective. Suppose that for all triples* (n, a_n, b_{n+1}) *with* $n \in \mathbb{N}$, $a_n \in A_n$, $b_{n+1} \in B_{n+1}$ *and* $f_n(a_n) = g_{n+1}^B(b_{n+1})$ *there exists* $a_{n+1} \in A_{n+1}$ *with* $f_{n+1}(a_{n+1}) = b_{n+1}$ *and* $g_{n+1}^A(a_{n+1}) = a_n$. *Then* $f : \varprojlim A_n \to \varprojlim B_n$ *is surjective.*

Proof. An element $b \in \varprojlim B_n$ is given by a sequence $(b_n)_{n \in \mathbb{N}}$ with $g_n^B(b_n) = b_{n-1}$. The hypothesis of the lemma allows us to inductively define a sequence $(a_n)_{n \in \mathbb{N}}$ such that $a_n \in A_n$, $g_n^A(a_n) = a_{n-1}$ and $f_n(a_n) = b_n$. The sequence $(a_n)_{n \in \mathbb{N}}$ defines an element of $\varprojlim A_n$ whose image under the limit map is b_n. □

Recall that the restriction of functions defines an inclusion $B^R_{[\frac{1}{q},\infty]} \subseteq B^R_{[1,\infty]}$ and Frobenius induces an isomorphism $\varphi\colon B^R_{[1,\infty]} \xrightarrow{\sim} B^R_{[\frac{1}{q},\infty]} \subseteq B^R_{[1,\infty]}$.

Lemma 5.19. *Let $k \in \mathbb{Z}_{\geq 0}$. The image of the map*

$$\psi_k\colon B^R_{[1,\infty]} \to B^R_{[1,\infty]}, \quad a \mapsto \pi^{-k}a - \varphi(a)$$

contains $[\varpi]B^R_{[1,\infty]}$. If $k > 0$, it contains $B^R_{[1,\infty]}$.

Proof. Let $A = \mathbb{W}(R^+)[\frac{\varpi}{\pi}]$. Recall that $B^R_{[1,\infty]} = A^\wedge_\pi[\frac{1}{\pi}]$. Thus, as $\psi_k(\pi^n x) = \pi^n\psi_k(x)$, it suffices to show that the image contains $[\varpi]A^\wedge_\pi$ (resp. $A^\wedge_\pi$ if $k > 0$). Let $x \in [\varpi]A^\wedge_\pi$ if $k = 0$ (resp. $x \in A^\wedge_\pi$ if $k > 0$). Note that the sequence $(\pi^{i\cdot k}\varphi^{(i-1)}(x))_{i\geq 1}$ in $A^\wedge_\pi$ converges π-adically to 0. (Use that $\varphi(A^\wedge_\pi) \subseteq A^\wedge_\pi$ and $\varphi([\varpi]) = [\varpi]^q$.) Thus $y = \sum_{i=1}^\infty \pi^{ik}\varphi^{(i-1)}(x)$ exists in $A^\wedge_\pi$. By π-adic continuity of Frobenius and hence of ψ_k, it is immediate that $\psi_k(y) = x$. ☐

Lemma 5.20. *Let $n \geq 1$ and let $A \in \mathrm{GL}_n(B^R_{[1,1]})$ be upper triangular with i-th diagonal entry π^{s_i} for some $s_i \in \mathbb{Z}$ (with $1 \leq i \leq n$). Assume that $s_1 \geq s_2 \geq \cdots \geq s_n$ holds. Then there exists a unipotent upper triangular matrix $U \in \mathrm{GL}_n(B^R_{[1,\infty]})$ such that $U^{-1}A\varphi(U)$ is upper triangular with entries in $B^R_{[0,1]}[\frac{1}{\pi}]$.*

Proof. We argue by induction on n. If $n = 1$, there is nothing to show. Assume n is fixed and we know the claim for all matrices of size $(n-1)\times(n-1)$. Let a_{ij} denote the (i,j)-th entry of A. Exploiting the induction hypothesis for the lower right $(n-1)\times(n-1)$-minor of A, we may assume that $a_{ij} \in B^R_{[0,1]}[\frac{1}{\pi}]$ for all $i > 1$. Let now $1 < j \leq n$. Suppose, by induction, that for all $1 < j' < j$, one has $a_{1j'} \in B_{[0,1]}[\frac{1}{\pi}]$. It suffices to find, in this situation, a unipotent upper triangular matrix $U \in \mathrm{GL}_n(B^R_{[1,\infty]})$ such that $U^{-1}A\varphi(U)$ has all the above properties of A and additionally its $(1,j)$-th entry lies in $B^R_{[0,1]}[\frac{1}{\pi}]$. Therefore, write $a_{1j} = a^{\mathrm{mer}}_{1j} + a'_{1j}$ with some $a^{\mathrm{mer}}_{1j} \in B^R_{[0,1]}[\frac{1}{\pi}]$ and $a'_{1j} \in [\varpi]B^R_{[1,\infty]}$, according to Lemma 5.17. By Lemma 5.19, there exists some $y \in B^R_{[1,\infty]}$ with $\psi_{s_1-s_j}(y) = a'_{1j}$ (here we use that $s_j \leq s_1$). Let $U = (U_{\ell m})_{\ell m} \in \mathrm{GL}_n(B^R_{[1,\infty]})$ be such that $U_{\ell m} = \delta_{\ell m}$ (where δ denotes the Kronecker-delta), unless $(\ell, m) = (1,j)$ and $U_{1j} = y$. Then it is immediate to compute that $U^{-1}A\varphi(U)$ satisfies all the claimed conditions. ☐

Proposition 5.21. *The forgetful functor $\mathcal{F}il^\gamma_{\mathrm{ss}} \to \mathfrak{B}^\diamondsuit$ factors through $(\mathfrak{B}^\diamondsuit)^{\mathrm{loc}}$ and defines a $\otimes$-exact equivalence*

$$\mathcal{F}il^\gamma_{\mathrm{ss}} \to (\mathfrak{B}^\diamondsuit)^{\mathrm{loc}}.$$

Proof. On points, any filtration splits since the category of isocrystals is semi-simple. In particular, the Newton polygon can be computed on the graded pieces. By the definition of semi-stable filtrations the Newton polygon is constant on the graded isocrystal.

We now prove fully-faithfulness. Let $\{\mathcal{E}_r\}_{r\in\mathbb{Q}}$ and $\{\mathcal{F}_r\}_{r\in\mathbb{Q}}$ be two semi-stable filtrations with underlying analytic isocrystals $\mathcal{E}$ and $\mathcal{F}$. Let $\mathcal{H}$ denote the $\underline{\mathrm{Hom}}$-bundle endowed with its induced semi-stable filtration $\{\mathcal{H}_r\}_{r\in\mathbb{Q}}$. We need to show that

$$\mathrm{Hom}_{\mathfrak{B}^\diamondsuit}(\mathcal{O}, \mathcal{H}) = \mathrm{Hom}_{\mathfrak{B}^\diamondsuit}(\mathcal{O}, \mathcal{H}_{\leq 0}).$$

We can prove inductively on the support of $\{\mathcal{H}_r\}_{r\in\mathbb{Q}}$ that

$$\mathrm{Hom}_{\mathfrak{B}^\diamondsuit}(\mathcal{O}, \mathcal{H}_{\leq r}/\mathcal{H}_{\leq 0}) = 0$$

for all $r > 0$ since the graded pieces all have slope larger than 0.

Since essential surjectivity can be proved v-locally by Lemma A.2, it suffices to show that the standard objects can be endowed with a semi-stable filtration, but this is clear. ☐

Proposition 5.22. *The forgetful functor* $\mathcal{F}il^{\mathrm{mer}}_{\mathrm{ss}} \to \mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}$ *factors through* $(\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}})^{\mathrm{loc}}$ *and defines a* $\otimes$-*exact equivalence*

$$\mathcal{F}il^{\mathrm{mer}}_{\mathrm{ss}} \to (\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}})^{\mathrm{loc}}\,.$$

Proof. It is automatic that the map respects the monoidal structure and exactness, since they are defined in terms of those of $\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}$. It follows from Proposition 5.21 that the map factors through $(\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}})^{\mathrm{loc}}$. To show fully-faithfulness, we may again pass to $\underline{\mathrm{Hom}}$-bundles $\mathcal{H}$ with semi-stable filtration $\{\mathcal{H}_r\}_{r\in\mathbb{Q}}$ as in the proof of Proposition 5.21. We need to show that

$$\mathrm{Hom}_{\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}}(\mathcal{O},\mathcal{H}) = \mathrm{Hom}_{\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}}(\mathcal{O},\mathcal{H}_{\leq 0}),$$

but as in the proof of Proposition 5.21 we can prove that $\mathrm{Hom}_{\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}}(\mathcal{O},\mathcal{H}_{\leq r}/\mathcal{H}_{\leq 0}) = 0$ for all $r > 0$.

Essential surjectivity can now be proved v-locally by Lemma A.2, hence it suffices to show that every isoshtuka $\mathcal{E} \in (\mathrm{Sht}_{\mathcal{Y}}[\frac{1}{\pi}])^{\mathrm{loc}}(S)$ can be endowed with a semi-stable filtration. In other words, we need to show that the unique semi-stable filtration of $\gamma(\mathcal{E})$ lifts to a filtration in $\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}$. Replacing E by its degree s field extension E_s, and since we have already proved fully-faithfulness, we may assume that the generic Newton polygon only takes values in $\mathbb{Z}$. Twisting by a line bundle, we may even assume that the smallest slope $\mathcal{E}$ is 0. We can now apply Proposition 5.9 and Lemma 5.10 to find a sub-bundle $\mathcal{O}^k \subseteq \mathcal{E}$, where k is the rank of $\gamma(\mathcal{E})_0$ and such that $\gamma(\mathcal{E})/\gamma(\mathcal{O}^k)$ has all slopes greater than 0. By induction on the rank, $\mathcal{E}/\mathcal{O}^k$ can be endowed with a semi-stable filtration $\{(\mathcal{E}/\mathcal{O}^k)_r\}_{r\in\mathbb{Q}}$ and we can lift this filtration to $\mathcal{E}$. □

6. G-BUNDLES WITH MEROMORPHIC STRUCTURE

6.1. **$\mathcal{G}$-structure.** Let $\mathcal{G}$ be a smooth affine group scheme over $\mathrm{Spec}\, O_E$. We denote by G its generic fiber over $\mathrm{Spec}\, E$, which we assume to be reductive. Later on, we will assume that $\mathcal{G}$ is a parahoric group scheme. We denote by $\mathrm{Rep}_{\mathcal{G}}$ (resp. Rep_G) the Tannakian category of algebraic representations of $\mathcal{G}$ over O_E (resp. of G over E).

Definition 6.1. We denote by $\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}} \in \mathcal{P}(\mathrm{PSch}^{\mathrm{aff}}, \mathrm{Grps})$ the presheaf valued in groupoids defined by

$$S \mapsto \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_{\mathcal{G}}, \mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}}(S))\,,$$

where $\mathrm{Fun}^{\otimes}_{\mathrm{ex}}$ denotes the $\otimes$-compatible O_E-linear exact functors. Analogously, we denote by $\mathfrak{B}(G) \in \mathcal{P}(\mathrm{PSch}^{\mathrm{aff}}, \mathrm{Grps})$ the presheaf valued in groupoids with

$$S \mapsto \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_{G}, \mathfrak{B}(S))\,.$$

Recall the loop group and positive loop group functors $LG, L^+\mathcal{G}\colon \mathrm{PSch}^{\mathrm{aff}} \to \mathrm{Sets}$ given on affine schemes $S = \mathrm{Spec}\, A$ by the formulas

$$LG(S) := G(\mathbb{W}(A)[\frac{1}{\pi}])$$

and

$$L^+\mathcal{G}(S) := \mathcal{G}(\mathbb{W}(A))\,.$$

We let LG and $L^+\mathcal{G}$ act on LG by φ-conjugation.

Proposition 6.2. *LG and $L^+\mathcal{G}$ are arc-sheaves.*

Proof. As both are ind-schemes and the arc-topology is subcanonical (in fact, canonical) on perfect $\mathbb{F}_p$-schemes by [BM21, Theorem 5.16], the claim follows. □

Proposition 6.3. *The following statements hold:*

(1) $\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}}$ *and* $\mathfrak{B}(G)$ *are scheme-theoretic small v-stacks.*
(2) The natural maps $LG \to \mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}}$ *and* $LG \to \mathfrak{B}(G)$ *are v-covers.*
(3) We have identities $\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}} = [LG /\!\!/_{\varphi} L^+\mathcal{G}]$ *and* $\mathfrak{B}(G) = [LG /\!\!/_{\varphi} LG]$.

Proof. The first statement follows from Proposition 4.5. Indeed, $\mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_{\mathcal{G}}, -)$ (resp. $\mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_{G}, -)$) is simply the mapping anima in the category $\mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,O_E}$ (resp. $\mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E}$) and this preserves v-sheaves.

To prove surjectivity of $LG \to \mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}}$ in the v-topology, it suffices by Remark 2.13 to show that for a product comb $\operatorname{Spec} R$, any $\mathcal{G}$-torsor on $\operatorname{Spec} \mathbb{W}R$ is trivial. Such a torsor becomes trivial after some étale cover of $\operatorname{Spec} \mathbb{W}R$, so it suffices to show that any étale cover of $\operatorname{Spec} \mathbb{W}R$ splits. As $\operatorname{Spec} \mathbb{W}R$ is henselian along the closed subscheme $\operatorname{Spec} R$, this follows from the same statement for $\operatorname{Spec} R$, which holds true by Remark 2.13. The surjectivity of $LG \to \mathfrak{B}(G)$ in the v-topology follows from Lemma 6.4 (see [Iva23, Theorem 6.1] for the case $G = \mathrm{GL}_n$). The last claim follows directly from the second one by computing the fiber products $LG \times_{\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}}} LG$ and $LG \times_{\mathfrak{B}(G)} LG$. ☐

The following slight generalization of [Ans22, Theorem 11.5] will be useful for our purposes.

Lemma 6.4. *If* $\operatorname{Spec}(A)$ *is a comb, then every* G*-torsor over* $\operatorname{Spec} \mathbb{W}(A)[\frac{1}{\pi}]$ *is trivial.*

Proof. We can follow the proof of [Ans22, Proposition 11.5], by noting that the reduction method in [Iva23, Section 6.1.1] (which is also used in [Ans22, Proposition 11.5]) works for general combs. ☐

Definition 6.5. We define the following four presheaves over $\mathrm{Perf}^{\mathrm{aff}}$ with values in groupoids:

(1) $\mathrm{Sht}_{\mathcal{Y},\mathcal{G}} \colon S \mapsto \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_{\mathcal{G}}, \mathrm{Sht}_{\mathcal{Y}}(S))$.
(2) $\mathrm{Isoc}_G \colon S \mapsto \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_{\mathcal{G}}, \mathfrak{B}^{\Diamond}(S))$.
(3) $\mathrm{Bun}^{\mathrm{mer}}_G \colon S \mapsto \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_{\mathcal{G}}, \mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}(S))$.
(4) $\mathrm{Sht}_{\mathbb{W},\mathcal{G}} \colon S \mapsto \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_{\mathcal{G}}, \mathrm{Sht}_{\mathbb{W}}(S))$.

Theorem 6.6. *The following statements hold:*

(1) $\mathrm{Sht}_{\mathcal{G}}$*,* $\mathrm{Sht}_{\mathbb{W},\mathcal{G}}$*,* Isoc_G *and* $\mathrm{Bun}^{\mathrm{mer}}_G$ *are small v-stacks.*
(2) We have a Cartesian diagram

$$\begin{tikzcd} \mathrm{Sht}_{\mathcal{G}} \arrow[r] \arrow[d] & \mathrm{Bun}^{\mathrm{mer}}_G \arrow[d] \\ \mathrm{Sht}_{\mathbb{W},\mathcal{G}} \arrow[r] & \mathrm{Isoc}_G \end{tikzcd} .$$

(3) We have identifications

$$\mathrm{Sht}_{\mathbb{W},\mathcal{G}} = (\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}})^{\Diamond} = [LG^{\Diamond} /\!\!/_{\varphi} L^{+}\mathcal{G}^{\Diamond}]$$

and

$$\mathrm{Isoc}_G = (\mathfrak{B}(G))^{\Diamond} = [LG^{\Diamond} /\!\!/_{\varphi} LG^{\Diamond}] .$$

(4) The maps $\mathrm{Sht}_{\mathbb{W},\mathcal{G}} \to \mathrm{Isoc}_G$ *and* $\mathrm{Sht}_{\mathcal{G}} \to \mathrm{Bun}^{\mathrm{mer}}_G$ *are v-covers.*

Proof. Since the application $\mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_{\mathcal{G}}, -)$ (resp. $\mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_{G}, -)$ commutes with 2-limits within $\mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,O_E}$ (resp. $\mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E}$) and all of $\mathrm{Sht}_{\mathcal{Y}}$, $(\mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}})^{\Diamond}$, $\mathfrak{B}^{\Diamond}$ and $\mathrm{Bun}^{\mathrm{mer}}_{\mathrm{FF}}$ are v-stacks in $\mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,O_E}$ or $\mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E}$, all of the presheaves of Definition 6.5 are v-sheaves. For the same reason, the second claim follows directly from Proposition 4.23. Furthermore, $\mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_{\mathcal{G}}, -)$ commutes with sheafification by Lemma A.8, which implies directly that $\mathrm{Sht}_{\mathbb{W},\mathcal{G}} = (\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}})^{\Diamond}$ and $\mathrm{Isoc}_G = \mathfrak{B}(G)^{\Diamond}$. Since the functor $(-)^{\Diamond}$ commutes with finite limits, it suffices to prove that the maps $LG^{\Diamond} \to \mathrm{Sht}_{\mathbb{W},\mathcal{G}}$ and $LG^{\Diamond} \to \mathrm{Isoc}_G$ are surjective to deduce the formulas from the third assertion. Let $\mathcal{F} \in \mathrm{Isoc}_G(S)$, the argument for $\mathrm{Sht}_{\mathbb{W},\mathcal{G}}$ being analogous. By part (1) of the theorem surjectivity can be shown v-locally, so we may assume $S = \operatorname{Spa}(R, R^+)$ is a product of points. By Proposition 4.14, we may even assume that for all objects $V \in \mathrm{Rep}_G$, the object $\mathcal{F}(V) \in \mathfrak{B}^{\Diamond}(S)$ is isomorphic to one in $\mathrm{Sht}_{\mathbb{W}}[\frac{1}{\pi}](S)$. We obtain a $\otimes$-exact functor from Rep_G to the category of projective $\mathbb{W}(R)[\frac{1}{\pi}]$-modules, which we interpret as a G-torsor over $\operatorname{Spec} \mathbb{W}(R)[\frac{1}{\pi}]$. By Lemma 6.4, such torsors are trivial over combs, and by Proposition 2.18, Spec R is a comb. After choosing a trivialization of $\mathcal{F}$, the φ-structure corresponds to an element $LG(\operatorname{Spec} R)$ which gives precisely a point $LG^{\lozenge}(S)$ lifting our original point. The final claim follows by base change from the third claim and the second claim. □

6.2. **Newton strata on** Isoc_G**.** We now wish to study the geometry of Isoc_G and $\mathrm{Bun}_G^{\mathrm{mer}}$. Recall the Kottwitz set $B(G)$, which classifies isocrystals with G-structure over algebraically closed fields, see [Kot97, §3]. Recall that $B(G)$ is naturally endowed with a partial order, see for example [RR96] or [Vie20, §3]. For $G = \mathrm{GL}_n$, $\mathcal{N}(G) = \mathcal{N}$ with $\mathcal{N}$ from Section 5.

Definition 6.7. Let $S = \operatorname{Spec} A \in \mathrm{PSch}^{\mathrm{aff}}$, and let $b \in B(G)$. We let $\mathfrak{B}(G)_{\leq b}(S) \subseteq \mathfrak{B}(G)(S)$ denote the full subcategory of objects $\mathcal{E} \in \mathfrak{B}(G)(S)$ whose Newton polygon is bounded by b at geometric points of S. We let $\mathfrak{B}(G)_b(S) \subseteq \mathfrak{B}(G)_{\leq b}(S)$ denote the full subcategory of objects $\mathcal{E} \in \mathfrak{B}(G)_{\leq b}(S)$ whose Newton polygon is exactly b at geometric points of S.

The following theorem due to work of various authors summarizes what we will need about the geometry of $\mathfrak{B}(G)$.

Theorem 6.8. *For any $b \in B(G)$ the map $\mathfrak{B}(G)_{\leq b} \to \mathfrak{B}(G)$ is a perfectly finitely presented closed immersion. Moreover, $\mathfrak{B}(G)_b = [* / \underline{G_b(\mathbb{Q}_p)}]$ as scheme-theoretic v-stacks.*

Proof. The first statement follows from [RR96, Theorem 3.6 (ii)], whose proof carries over the characteristic p setting, see [HV11, Theorem 7.3]. The last statement follows from [HK22, Theorem 2.11]. □

Proposition 6.9. *The elements of $|\mathrm{Isoc}_G|$ are in bijection with $B(G)$.*

Proof. By definition, points in $|\mathrm{Isoc}_G|$ are in bijection with equivalence classes of $\mathrm{Spa}(C, C^+)$-valued points of Isoc_G. By Lemma 5.3, these are the same as isocrystals with G-structure which are classified by $B(G)$, see [Kot97, §3]. □

Definition 6.10. Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}^{\mathrm{aff}}$. We let $\mathrm{Isoc}_G^{\leq b}(S) \subseteq \mathrm{Isoc}_G(S)$ denote the full subcategory of objects $\mathcal{E} \in \mathrm{Isoc}_G(S)$ whose Newton polygon is bounded by b at geometric points of S. We let $\mathrm{Isoc}_G^b(S) \subseteq \mathrm{Isoc}_G^{\leq b}(S)$ denote the full subcategory of objects $\mathcal{E} \in \mathrm{Isoc}_G^{\leq b}(S)$ whose Newton polygon is exactly b at geometric points of S.

Proposition 6.11. *For any $b \in B(G)$, the map $\mathrm{Isoc}_G^{\leq b} \to \mathrm{Isoc}_G$ is a closed immersion and agrees with $\mathfrak{B}(G)_{\leq b}^{\lozenge}$. The map $\mathrm{Isoc}_G^b \to \mathrm{Isoc}_G^{\leq b}$ is an open immersion. Moreover, $\mathrm{Isoc}_G^b = \mathfrak{B}(G)_b^{\lozenge} = [* / \underline{G_b(\mathbb{Q}_p)}]$ as v-stacks.*

Proof. Since $\lozenge$ preserves open and closed immersions, it suffices to identify $\mathrm{Isoc}_G^{\leq b}$ and Isoc_G^b with $\mathfrak{B}(G)_{\leq b}^{\lozenge}$ and $\mathfrak{B}(G)_b^{\lozenge}$, respectively. Let $S = \mathrm{Spa}(R, R^+)$. By definition, $\mathrm{Isoc}_G^{\leq b}(S)$ is the subcategory of objects $\mathcal{E} \in \mathrm{Isoc}_G(S)$ whose Newton polygon is pointwise bounded by b at every geometric point S. On the other hand, $\mathfrak{B}(G)_{\leq b}^{\lozenge_{\mathrm{pre}}}(S)$ corresponds to G-isocrystals over Spec R whose polygon is bounded by b at every geometric point of Spec R. To prove $\mathfrak{B}(G)_{\leq b}^{\lozenge} = \mathrm{Isoc}_G^{\leq b}$, it suffices to show that v-locally having a Newton polygon bounded by b for $\mathrm{Spa}(R, R^+)$ or for Spec R agree. Of course, the schematic condition is stronger than the analytic one, since on the analytic side a condition is imposed only on those ideals of Spec R that support a continuous valuation. Now, over product of points the two conditions agree. Indeed, principal connected components of a product of points support a continuous valuation. Moreover, these components are dense in Spec R.

A similar argument shows $\mathfrak{B}(G)_b^{\lozenge} = \mathrm{Isoc}_G^b$. Indeed, if S is a product of points, all of the maximal ideals of Spec R support a continuous valuation and the map $\mathfrak{B}(G)_b \to \mathfrak{B}(G)_{\leq b}$ is open.

The last claim follows directly from Proposition 2.19. □

6.3. **Newton strata on** $\mathrm{Bun}_G^{\mathrm{mer}}$**.** Recall the moduli stack $\mathcal{M}$ of [FS24, Definition V.3.2]. The connected components of $\mathcal{M}$ are indexed by $b \in B(G)$ and the maps $\mathcal{M}_b \to \mathrm{Bun}_G$ are the so-called smooth charts.

Proposition 6.12. *The v-stack $\mathcal{M}$ is the moduli stack given by the formula*

$$\mathcal{M}\colon\ S \mapsto \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_G, \mathcal{F}il^{\sigma}_{\mathrm{ss}}(S))\,.$$

Proof. It follows directly from the definition. ☐

Theorem 6.13. *The moduli stack $\mathcal{M}$ fits in the Cartesian diagram*

$$\begin{tikzcd} \mathcal{M} \arrow[r] \arrow[d] & \mathrm{Bun}_G^{\mathrm{mer}} \arrow[d, "\gamma"] \\ \coprod_{b\in B(G)} \mathrm{Isoc}_G^b \arrow[r] & \mathrm{Isoc}_G \end{tikzcd}$$

of small v-stacks.

Remark 6.14. While this article was in preparation, we learned from a private communication with Z. Wu that he had proven independently a version of Theorem 6.13 in the language of relative Robba rings.

Proof. Observe that we have the identification

$$\coprod_{b\in B(G)} \mathrm{Isoc}_G^b(S) = \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_G, (\mathfrak{B}^{\Diamond})^{\mathrm{loc}}(S))\,.$$

Since $\mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_G, -)$ commutes with limits, it suffices to show that $\mathcal{F}il^{\sigma}_{\mathrm{ss}}(S)$ fits in the Cartesian diagram

$$\begin{tikzcd} \mathcal{F}il^{\sigma}_{\mathrm{ss}}(S) \arrow[r] \arrow[d] & \mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S) \arrow[d] \\ (\mathfrak{B}^{\Diamond})^{\mathrm{loc}}(S) \arrow[r] & \mathfrak{B}^{\Diamond}(S)\,. \end{tikzcd}$$

By definition, $(\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}})^{\mathrm{loc}}$ fits as the upper-left entry of the above Cartesian diagram. By Proposition 5.16 and Proposition 5.22,

$$(\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}})^{\mathrm{loc}}(S) \cong \mathcal{F}il^{\mathrm{mer}}_{\mathrm{ss}} \cong \mathcal{F}il^{\sigma}_{\mathrm{ss}}(S)\,.$$

☐

Corollary 6.15. *Let $S = \mathrm{Spa}(R, R^+)$ and let $\mathcal{E} \in \mathrm{Bun}_{\mathrm{FF}}(S)$. The following hold:*

(1) After replacing S by a v-cover, $\mathcal{E}$ can be lifted to $\mathrm{Bun}_{\mathrm{FF}}^{\mathrm{mer}}(S)$.
(2) After replacing S by a v-cover, $\mathcal{E}$ can be lifted to $\mathrm{Sht}_{\mathcal{Y}}(S)$.
(3) The map of small v-stacks $\mathrm{Bun}_G^{\mathrm{mer}} \to \mathrm{Bun}_G$ is surjective.
(4) The map of small v-stacks $\mathrm{Sht}_G \to \mathrm{Bun}_G$ is surjective.

Proof. The first and second claims are particular instances of the third and fourth claim in the case where $G = \mathrm{GL}_n$. For the third claim, the map $\mathcal{M} \to \mathrm{Bun}_G$ is formally and ℓ-cohomologically smooth and surjects onto its image. In particular, it is a surjection of small v-stacks. The result follows since this map factors through $\mathrm{Bun}_G^{\mathrm{mer}} \to \mathrm{Bun}_G$. The fourth claim follows from Theorem 6.6 and the third claim. ☐

Definition 6.16. Given two subsets $U_1, U_2 \subseteq B(G)$ we let $\mathcal{M}^{\sigma\in U_2}_{\gamma\in U_1}$ denote $\gamma^{-1}(\mathrm{Isoc}_G^{U_1}) \cap \sigma^{-1}(\mathrm{Bun}_G^{U_2})$. Whenever $U_i = B(G)$, we omit the subscript or superscript as an abbreviation.

We will mostly use Definition 6.16 when U_1 or U_2 are given by Newton polygon inequalities. In this case, we use more intuitive notation: for example, $\mathcal{M}^{\sigma=b} := \sigma^{-1}(\mathrm{Bun}_G^b)$ and $\mathcal{M}_{\gamma=b} := \gamma^{-1}(\mathrm{Isoc}_G^b) = \mathcal{M}_b$.

7. Three comparison theorems

7.1. **The meromorphic comparison.** Let C be a non-Archimedean algebraically closed field and let $S = \mathrm{Spa}(C, O_C)$. One interesting consequence of the classification theorem of vector bundles on the Fargues–Fontaine curve is that every such vector bundle extends at ∞, i.e. it is isomorphic to one obtained from a φ-module over $Y_{(0,\infty],S}$ for such S. In what follows, we will prove that this statement holds in families when one is allowed to work v-locally.

Definition 7.1. Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}^{\mathrm{aff}}$ and $T = \mathrm{Spa}(R, R^\circ)$.

1. We let $\mathrm{Bun}^+_{\mathrm{FF}}(S) \in \mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E})$ be given by the rule that attaches to S the category of pairs $(\mathcal{E}, \Phi)$, where $\mathcal{E}$ is a vector bundle over $Y_{(0,\infty],T}$ and $\Phi\colon\ \varphi^*\mathcal{E} \to \mathcal{E}$ is an isomorphism.
2. We say that $\mathcal{E} \in \mathrm{Bun}_{\mathrm{FF}}(S)$ *extends* at ∞ if it is in the essential image of the map $\mathrm{Bun}^+_{\mathrm{FF}}(S) \to \mathrm{Bun}_{\mathrm{FF}}(T) \cong \mathrm{Bun}_{\mathrm{FF}}(S)$.
3. We denote by $(\mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}})^{\dagger_{\mathrm{pre}}} \in \mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,O_E})$, the presheaf given by the rule
$$(R, R^+) \mapsto \mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}}(\mathrm{Spec}\, R^\circ).$$
4. We say that $\mathcal{E} \in \mathrm{Sht}_{\mathcal{Y}}(S)$ is a *BKF-shtuka* if it is in the essential image of the map $(\mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}})^{\dagger_{\mathrm{pre}}}(S) \to \mathrm{Sht}_{\mathcal{Y}}(T) \cong \mathrm{Sht}_{\mathcal{Y}}(S)$.
5. We denote by $(\mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}])^{\dagger_{\mathrm{pre}}} \in \mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E})$, the presheaf given by the rule
$$(R, R^+) \mapsto \mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}](\mathrm{Spec}\, R^\circ)\,.$$
6. We denote by $\mathfrak{B}^{\dagger_{\mathrm{pre}}} \in \mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E})$, the presheaf given by the rule
$$(R, R^+) \mapsto \mathfrak{B}(\mathrm{Spec}\, R^\circ)\,.$$

We can pass to $\mathcal{G}$-objects for all the above.

Definition 7.2. Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}^{\mathrm{aff}}$ and $T = \mathrm{Spa}(R, R^\circ)$.

1. We let $\mathrm{Bun}^+_G(S) \in \mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Grps})$ be given by $\mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_G, \mathrm{Bun}^+_{\mathrm{FF}})$.
2. We say that $\mathcal{E} \in \mathrm{Bun}_G(S)$ *extends* at ∞ if it is in the essential image of the map $\mathrm{Bun}^+_G(S) \to \mathrm{Bun}_G(S)$.
3. We denote $(\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}})^{\dagger_{\mathrm{pre}}} = \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_{\mathcal{G}}, (\mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}})^{\dagger_{\mathrm{pre}}})$.
4. We say that $\mathcal{E} \in \mathrm{Sht}_{\mathcal{G}}(S)$ is a *BKF-$\mathcal{G}$-shtuka* if it is in the essential image of the map $(\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}})^{\dagger_{\mathrm{pre}}}(S) \to \mathrm{Sht}_{\mathcal{G}}(S)$.
5. We let $(\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}}[\frac{1}{\pi}])^{\dagger_{\mathrm{pre}}} \in \mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Grps})$ be given by $\mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_G, (\mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}}[\frac{1}{\pi}])^{\dagger_{\mathrm{pre}}})$.
6. We let $\mathfrak{B}(G)^{\dagger_{\mathrm{pre}}} \in \mathcal{P}(\mathrm{Perf}^{\mathrm{aff}}, \mathrm{Grps})$ be given by $\mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_G, (\mathfrak{B})^{\dagger_{\mathrm{pre}}})$.

Proposition 7.3. *Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}^{\mathrm{aff}}$, the following hold.*

1. *The map $\mathrm{Bun}^+_{\mathrm{FF}}(S) \to \mathrm{Bun}_{\mathrm{FF}}(S)$ is exact and fully-faithful.*
2. *The map $\mathrm{Bun}^+_G(S) \to \mathrm{Bun}_G(S)$ is fully-faithful.*
3. *If S is a product of points, the diagrams*
$$\begin{tikzcd}
(\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}})^{\dagger_{\mathrm{pre}}}(S) \arrow[r] \arrow[d] & \mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}}[\frac{1}{\pi}]^{\dagger_{\mathrm{pre}}}(S) \arrow[r] \arrow[d] & \mathrm{Bun}^+_G(S) \arrow[d] \\
\mathrm{Sht}_{\mathcal{G}}(S) \arrow[r] & \mathrm{Sht}_{\mathcal{G}}[\frac{1}{\pi}](S) \arrow[r] & \mathrm{Bun}_G(S)
\end{tikzcd}$$
are Cartesian in Grps.
4. *If S is a product of points then $\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}}[\frac{1}{\pi}]^{\dagger_{\mathrm{pre}}}(S) \cong \mathfrak{B}(G)^{\dagger_{\mathrm{pre}}}(S)$.*

(5) *The sheafification of* $\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}}[\frac{1}{\pi}]^{\dagger_{\mathrm{pre}}}$ *is* $\mathfrak{B}(G)^{\dagger}$.

Proof. The first claim is [PR24, Proposition 2.1.3], whose proof generalizes to general E (see [PR24, Remark 2.1.10]). The second claim follows formally by passing to $\mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_G, -)$. For the third claim, note that by Kedlaya's GAGA [Ked20, Theorem 3.8], we can identify the category $\mathrm{Sht}_{\mathcal{G}}(S) \times_{\mathrm{Bun}_G(S)} \mathrm{Bun}^{+}_{\mathcal{G}}(S)$ with the category of $\mathcal{G}$-bundles over $\mathrm{Spec}\,\mathbb{W}(R^{\circ}) \setminus (\{\pi = 0\} \cap \{[\varpi] = 0\})$ together with φ-action defined over $\mathrm{Spec}\,\mathbb{W}(R^{\circ})[\frac{1}{\pi}]$. We note that to carry [Ked20, Theorem 3.8] over to the equal characteristic setting, it suffices to generalize [Ked20, Proposition 3.6] to ramified Witt vectors, which is straightforward since the proof strategy works in this generality. As S is a product of points, by [Ans22, Theorem 1.1]) (see [Gle21, Proposition 2.1.17]), any such $\mathcal{G}$-bundle extends uniquely to a $\mathcal{G}$-bundle over $\mathrm{Spec}\,\mathbb{W}(R^{\circ})$. This proves that the outer diagram is Cartesian. Moreover, the same argument also applies to the right square, so this is Cartesian as well. It then follows that the left square is Cartesian.

For the fourth claim, write $S = \mathrm{Spa}(R, R^{+})$. In the case that $\mathcal{G} = \mathrm{GL}_n$, we need to show that any isocrystal $\mathcal{E}$ over $\mathbb{W}(R^{\circ})[\frac{1}{\pi}]$ contains a $\mathbb{W}(R^{\circ})$-lattice. As S is a product of points, Proposition 2.18 and [Iva23, Theorem 6.1] imply that $\mathcal{E}$ is free as a $\mathbb{W}(R^{\circ})[\frac{1}{\pi}]$-module, but then an $\mathbb{W}(R^{\circ})$-lattice obviously exists. For more general $\mathcal{G}$ this follows from Lemma 6.4. The fifth claim follows from the fourth since product combs are a basis for the v-topology. □

Remark 7.4. We warn the reader that the maps $(\mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}})^{\dagger_{\mathrm{pre}}}(S) \to \mathrm{Sht}_{\mathcal{Y}}(S)$ and $\mathrm{Bun}^{+}_{\mathrm{FF}}(S) \to \mathrm{Bun}_{\mathrm{FF}}(S)$ do not reflect exactness. For this reason one crucially relies on the in-depth analysis of $\mathcal{G}$-torsors for parahoric $\mathcal{G}$, see [Ans22, Theorem 1.1].

The advantage of working with $(\mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}})^{\dagger_{\mathrm{pre}}}$ is that its values on product of points are easy to describe.

Proposition 7.5. *Let* $S = \mathrm{Spa}(R, R^{+})$ *be a product of points with* $R^{+} = R^{\circ} = \prod_{i\in I} O_{C_i}$*, then the restriction functor*

$$(\mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}})^{\dagger_{\mathrm{pre}}}(S) \to \prod_{i\in I} \mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}}(\mathrm{Spec}\, O_{C_i})$$

is fully-faithful, and its essential image is the collection of families of $\{(\mathcal{E}_i, \Phi_i)\}_{i\in I}$ *with uniformly bounded zeros and poles on* π.

Proof. The fully-faithful functor is induced by the isomorphism $\mathbb{W}(\prod O_{C_i}) = \prod \mathbb{W}(O_{C_i})$. The pole (resp. zero) at each $i \in I$ of any object in the essential image is bounded by the pole (resp. zero) of its preimage. Conversely, if we have a uniform bound, then the Frobenius is represented by a matrix with entries in $\mathbb{W}(R^{\circ})[\frac{1}{\pi}] = (\prod \mathbb{W}(O_{C_i}))[\frac{1}{\pi}] \subseteq \prod(\mathbb{W}(O_{C_i})[\frac{1}{\pi}])$, whose inverse also has entries in this subring. □

Moreover, at the level of geometric points $\mathrm{Sht}_{\mathcal{Y}}$ is also easy to describe. Indeed, the following is the $\pi = \xi$ version of Fargues' theorem [SW20, Theorem 14.1.1].

Proposition 7.6. *Let C be a non-Archimedean field, then the following categories are equivalent:*

(1) *BKF-modules with* $\xi = \pi$*. In other words, the category pairs* (M, Φ)*, where M is a free* $\mathbb{W}(O_C)$*-module and* $\Phi\colon M[\frac{1}{\pi}] \to \mathbb{W}(O_C)_{\varphi} \otimes_{\mathbb{W}(O_C)} M[\frac{1}{\pi}]$ *is an isomorphism.*
(2) $(\mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}})^{\dagger_{\mathrm{pre}}}(C, C^{+})$
(3) $\mathrm{Sht}_{\mathcal{Y}}(C, C^{+})$.

Proof. By definition $(\mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}})^{\dagger_{\mathrm{pre}}}(C, C^{+}) = \mathrm{Sht}^{\mathrm{sch}}_{\mathbb{W}}(O_C)$, which is precisely the category of BKF-modules with $\xi = \pi$, so the first two categories are the same category. The equivalence with the third category is given in [SW20, §12-14] when $\xi \neq \pi$. The same proof strategy applies and generalizes to general E (see the proof of Proposition 7.3). □

We wish to extend Proposition 7.6 to the case of product of points. It will be profitable to work with the stack of shtukas that have their leg away from the trivial untilt. Let us set some notation.

We let $S \in \mathrm{Perf}^{\mathrm{aff}}$ of the form $S = \mathrm{Spa}(R, R^+)$. Recall that an untilt $\infty : S \to \mathrm{Spd}\, O_E$ is given by a degree 1 Cartier divisor $\infty : S^\sharp \hookrightarrow \mathcal{Y}_S$, and that it factors through $\mathrm{Spd}\, E$ if and only if the Cartier divisor factors through Y_S. In this circumstance, we let $\xi_{S^\sharp} \in \mathbb{W}(R^+)$ be a generator of the kernel of $\mathbb{W}(R^+) \to R^{\sharp,+}$, cf. [SW20, Proposition 11.3.1] and [FS24, Proposition II.1.4].

Definition 7.7. Let $S \in \mathrm{Perf}^{\mathrm{aff}}$ be of the form $S = \mathrm{Spa}(R, R^+)$.

(1) A *$\mathcal{G}$-shtuka* is a triple $(S^\sharp, \mathcal{E}, \Phi)$, where $\infty : S^\sharp \hookrightarrow \mathcal{Y}_S$ is an untilt of S over $\mathrm{Spd}(O_E)$, $\mathcal{E}$ is a $\mathcal{G}$-bundle over $\mathcal{Y}_S$ and $\Phi_{\mathcal{E}}$ is an isomorphism
$$\Phi_{\mathcal{E}} : (\varphi^*\mathcal{E})_{\mathcal{Y}_S \setminus \infty} \to \mathcal{E}_{\mathcal{Y}_S \setminus \infty}$$
that is meromorphic (cf. [SW20, Definition 5.3.5]) along $\infty = \{\xi = 0\}$.

(2) A *BKF-$\mathcal{G}$-shtuka* with leg at ∞ is a triple $(S^\sharp, \mathcal{E}, \Phi)$, where $\mathcal{E}$ is a $\mathcal{G}$-bundle over $\mathrm{Spec}\, \mathbb{W}(R^\circ)$ and $\Phi_{\mathcal{E}}$ is an isomorphism
$$\Phi_{\mathcal{E}} : (\varphi^*\mathcal{E}) \to \mathcal{E}$$
defined over $\mathrm{Spec}\, \mathbb{W}(R^\circ)[\frac{1}{\xi_{S^\sharp}}]$.

(3) We let $\mathrm{Sht}_{\mathcal{G},O_E}$ denote the v-stack of $\mathcal{G}$-shtukas and we let $\mathrm{Sht}^+_{\mathcal{G},O_E}$ denote the prestack of BKF-$\mathcal{G}$-shtukas.

Observe that we have a map
$$\mathrm{Sht}_{\mathcal{G},O_E} \to \mathrm{Bun}_G$$
that sends $(S^\sharp, \mathcal{E}, \Phi)$ to the unique $\mathcal{G}$-φ-module over Y_S that agrees with $(\mathcal{E}, \Phi)$ over $Y_{[r,\infty),S}$ for sufficiently large r.

Proposition 7.8. *If S is a product of points we have a Cartesian diagram*
$$\begin{tikzcd} \mathrm{Sht}^+_{\mathcal{G},O_E}(S) \arrow[r] \arrow[d] & \mathrm{Bun}^+_G(S) \arrow[d] \\ \mathrm{Sht}_{\mathcal{G},O_E}(S) \arrow[r] & \mathrm{Bun}_G(S) \end{tikzcd}$$
in Grps.

Proof. The same proof as in Proposition 7.3 works in this generality. □

Recall that the stack of shtukas $\mathrm{Sht}_{\mathcal{G},E}$ (i.e. the locus in $\mathrm{Sht}_{\mathcal{G},O_E}$ where the untilt $\infty : S \to \mathrm{Spd}\, O_E$ factors through $\mathrm{Spd}\, E$) admits a different description.

Proposition 7.9. *Let $S \in \mathrm{Perf}^{\mathrm{aff}}$ and consider the groupoid $\mathrm{Sht}_{\mathcal{G},E}(S)$. It is equivalent to the groupoid of tuples $(S^\sharp, T, \mathcal{F}, \alpha)$, where $S^\sharp$ is an untilt of S over E, T is a quasi-pro-étale $\underline{\mathcal{G}(O_E)}$-torsor over S, $\mathcal{F} \in \mathrm{Bun}_G(S)$ and $\alpha : \mathcal{E}_T \to \mathcal{F}$ is a modification over $X_{\mathrm{FF},S}$, where $\mathcal{E}_T \in \mathrm{Bun}^1_G(S)$ is the unique G-bundle over $X_{\mathrm{FF},S}$ specified by the identification $\mathrm{Bun}^1_G \simeq [* / \underline{G(E)}]$. Under this equivalence the map $\mathrm{Sht}_{\mathcal{G},E} \to \mathrm{Bun}_G$ is given by*
$$(S^\sharp, T, \mathcal{F}, \alpha) \mapsto \mathcal{F}\,.$$
Moreover, the Beauville–Laszlo uniformization map $\mathrm{BL} : \mathrm{Gr}_{G,E} \to \mathrm{Bun}_G$ factors as $\mathrm{Gr}_{G,E} \to \mathrm{Sht}_{\mathcal{G},E} \to \mathrm{Bun}_G$.

Proof. This is [Zha23, Proposition 11.16], whose proof carries over to general E. □

Lemma 7.10. *Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}^{\mathrm{aff}}$ be a product of points with $R^+ = \prod_{i \in I} C_i^+$ and pseudo-uniformizer $\varpi \in R^+$. Then $\mathrm{Sht}^+_{\mathcal{G},E}(S) \to \mathrm{Sht}_{\mathcal{G},E}(S)$ is an equivalence.*

Proof. By Proposition 7.8 the map is fully-faithful. We wish to show it is essentially surjective. Fix $(S^\sharp, \mathcal{E}, \Phi) \in \mathrm{Sht}_{\mathcal{G},E}(S)$. From now on we will omit the untilt from the notation. After fixing an embedding $\mathcal{G} \to \mathrm{GL}_n$, we see that Φ has uniformly bounded zeroes and poles along ∞.

Let $S_i = \mathrm{Spa}(C_i, C_i^+)$. Over geometric points, $\mathrm{Sht}^+_{\mathcal{G},E}(S_i) \simeq \mathrm{Sht}_{\mathcal{G},E}(S_i)$ by [Ans22, Theorem 1.1]. Let $(\mathcal{E}_i, \Phi_i)$ be the BKF-$\mathcal{G}$-shtuka obtained from $(\mathcal{E}, \Phi)$ after restricting to S_i. We may find a trivialization $\mathcal{G} \simeq \mathcal{E}_i$ and by transfer of structure obtain a matrix $M_i \in \mathcal{G}(\mathbb{W}(O_{C_i})[\frac{1}{\xi}])$. We consider the product BKF-$\mathcal{G}$-shtuka $(\mathcal{E}_\infty, \Phi_\infty)$ obtained from the product matrix $M_\infty = \prod_{i\in I} M_i \in \prod_{i\in I} \mathcal{G}(\mathbb{W}(O_{C_i})[\frac{1}{\xi}])$. Since we have uniform bounds on poles and zeroes of ξ, this matrix lies in $\mathcal{G}([\prod_{i\in I} \mathbb{W}(O_{C_i})][\frac{1}{\xi}])$.

We claim that $(\mathcal{E}_\infty, \Phi_\infty)$ is isomorphic to $(\mathcal{E}, \Phi)$. Let $\mathcal{I}$ denote the groupoid of isomorphisms between $(\mathcal{E}, \Phi)$ and $(\mathcal{E}_\infty, \Phi_\infty)$, we claim that $\mathcal{I}$ is proper and quasi-pro-étale over S.

Indeed, we reinterpret both shtukas in terms of Proposition 7.9, so that we have tuples $(T, \mathcal{F}, \alpha)$ and $(T_\infty, \mathcal{F}_\infty, \alpha_\infty)$. An isomorphism of this data consists of a pair (Θ_1, Θ_2), where $\Theta_1 : T \to T_\infty$ is an $\underline{\mathcal{G}(O_E)}$-equivariant isomorphism while $\Theta_2 : \mathcal{F} \to \mathcal{F}_\infty$ is an isomorphism in $\mathrm{Bun}_G(S)$. This data should fit in the following commutative diagram

$$\begin{tikzcd} \mathcal{E}_T \arrow[r, "\Theta_1"] \arrow[d, dashed, "\alpha"] & \mathcal{E}_{T_\infty} \arrow[d, dashed, "\alpha_\infty"] \\ \mathcal{F} \arrow[r, "\Theta_2"] & \mathcal{F}_\infty, \end{tikzcd}$$

where the solid arrows denote maps over X_{FF} while the dashed arrows denote modifications. Notice that Θ_1 determines Θ_2 (if they exist) and vice versa.

We make two observations. Since S is a product of points, every quasi-pro-étale $\underline{\mathcal{G}(O_E)}$-torsor is trivial by Lemma 2.16. This implies that the space of isomorphism between T and T_∞ is isomorphic to $\underline{\mathcal{G}(O_E)}$ and in particular it is proper and quasi-pro-étale over S. The second observation is that any isomorphism $\Theta_1 : T \to T_\infty$ gives rise to a unique commutative diagram

$$\begin{tikzcd} \mathcal{E}_T \arrow[r, "\Theta_1"] \arrow[d, dashed, "\alpha"] & \mathcal{E}_{T_\infty} \arrow[d, dashed, "\alpha_\infty"] \\ \mathcal{F} \arrow[r, dashed, "\Theta_2"] & \mathcal{F}_\infty. \end{tikzcd}$$

Indeed, $\Theta_2 = \alpha_\infty \circ \Theta_1 \circ \alpha^{-1}$. Moreover, after fixing trivializations $\underline{\mathcal{G}(O_E)} \simeq T$ and $\underline{\mathcal{G}(O_E)} \simeq T_\infty$, we have a map

$$\underline{\mathcal{G}(O_E)} \to \mathrm{Hck}^{\mathrm{loc}}_{\mathcal{G},S^\sharp} \qquad \text{with} \qquad \Theta_1 \mapsto [\mathcal{F}^{\mathrm{loc}} \xrightarrow{\Theta_2} \mathcal{F}^{\mathrm{loc}}_\infty]$$

to the local Hecke stack of [FS24, Definition VI.1.6]. Θ_1 defines an isomorphism if and only if the induced map $\mathcal{F}^{\mathrm{loc}} \xrightarrow{\Theta_2} \mathcal{F}^{\mathrm{loc}}_\infty$ is an isomorphism instead of merely a modification. Since this is a closed condition within the local Hecke stack, the v-sheaf $\mathcal{I}$ of isomorphisms between $(\mathcal{E}, \Phi)$ and $(\mathcal{E}_\infty, \Phi_\infty)$ is a closed subsheaf of $\underline{\mathcal{G}(O_E)}$. In particular, $\mathcal{I}$ is proper and quasi-pro-étale over S.

To show that $(\mathcal{E}_\infty, \Phi_\infty)$ is isomorphic to $(\mathcal{E}, \Phi)$ it suffices to show that $\mathcal{I} \to S$ admits a section. We note that it is v-surjective since the map is qcqs and $|\mathcal{I}| \to |S|$ is surjective. Indeed, it is a closed map which by construction maps to every principal component of $\pi_0(S)$ and these are dense in S. Using Lemma 2.16 again, we observe that the map $\mathcal{I} \to S$ admits a section. $\square$

Corollary 7.11. *Let $S \in \mathrm{Perf}^{\mathrm{aff}}$ be a product of points. Then, the following statements hold:*

(1) The map $\mathrm{Bun}^+_G(S) \to \mathrm{Bun}_G(S)$ is an equivalence.

(2) The map $\mathrm{Sht}^+_{\mathcal{G},O_E}(S) \to \mathrm{Sht}_{\mathcal{G},O_E}(S)$ *is an equivalence of groupoids.*

Proof. In light of Proposition 7.8 it suffices to show the first statement. Recall from [FS24, §III.3] the Beauville–Laszlo uniformization map

$$\mathrm{BL} : \mathrm{Gr}_{G,E} \to \mathrm{Bun}_G$$

$$\mathrm{BL}(\alpha_{\mathrm{dR}} : \mathcal{E}_{\mathrm{dR}} \to G) := \alpha_{X_{\mathrm{FF}}} : \mathcal{E}_{X_{\mathrm{FF}}} \to G$$

that sends a modification of the trivial G-torsor over B_{dR} to a modification of the trivial G-torsor over X_{FF}. This map is a pro-étale surjection by [FS24, Proposition III.3.1].

Recall that under the notation of Proposition 7.9, this map factors as

$$\mathrm{Gr}_{G,E} \to \mathrm{Sht}_{\mathcal{G},E} \to \mathrm{Bun}_G\,.$$

By Lemma 2.16, any map $S \to \mathrm{Bun}_G$ lifts to a map $S \to \mathrm{Gr}_{G,E}$ which induces $S \to \mathrm{Sht}_{\mathcal{G},E}$. By Lemma 7.10, this comes from an object in $\mathrm{Sht}^+_{\mathcal{G},E}$ which induces an element in Bun^+_G as we wanted to show. □

Theorem 7.12. *The following statements hold:*

(1) We have an isomorphism of small v-stacks $(\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}})^\dagger \cong \mathrm{Sht}_{\mathcal{G}}$.

(2) We have an isomorphism of small v-stacks $\mathfrak{B}(G)^\dagger \cong \mathrm{Bun}^{\mathrm{mer}}_G$.

(3) The maps $(\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}})^\diamond \to \mathrm{Sht}_{\mathcal{G}}$ *and* $\mathfrak{B}(G)^\diamond \to \mathrm{Bun}^{\mathrm{mer}}_G$ *are v-surjective.*

Proof. Proposition 7.3 shows that $(\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}})^{\dagger\mathrm{pre}}(S) \to \mathrm{Sht}_{\mathcal{G}}(S)$ is fully-faithful and Corollary 7.11 shows that it is v-locally surjective since it is an equivalence on products of points. This proves the first claim. For the second claim, consider the map

$$\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}}[\frac{1}{\pi}]^{\dagger\mathrm{pre}}(S) \to \mathrm{Sht}_{\mathcal{G}}[\frac{1}{\pi}](S)\,.$$

Arguing as above, we see that it is fully-faithful in general and an equivalence when S is a product of points. In particular, their v-sheafifications agree. Nevertheless, by the last part of Proposition 7.3, the v-sheafification of $\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}}[\frac{1}{\pi}]^{\dagger\mathrm{pre}}$ is $\mathfrak{B}(G)^\dagger$ while the v-sheafification of $\mathrm{Sht}_{\mathcal{G}}[\frac{1}{\pi}]$ is $\mathrm{Bun}^{\mathrm{mer}}_G$.

For the third claim, it suffices to prove that $(\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}})^\diamond \to \mathrm{Sht}_{\mathcal{G}}$ is surjective since $\mathrm{Sht}_{\mathcal{G}} \to \mathrm{Bun}^{\mathrm{mer}}_G$ is surjective and the map $(\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}})^\diamond \to \mathrm{Bun}^{\mathrm{mer}}_G$ factors through $\mathfrak{B}(G)^\diamond$. Since $\mathrm{Sht}_{\mathcal{G}} \cong (\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}})^\dagger$, it suffices to prove that $\mathcal{E} \in (\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}})^{\dagger\mathrm{pre}}(S)$ lifts to an object in $(\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}})^{\diamond\mathrm{pre}}(S')$ for some v-cover $S' \to S$. We can reduce this to the case where $S = \mathrm{Spa}(R, R^+)$ is a product of points with $R^+ = \prod_{i\in I} C_i^+$, and $\mathcal{E}$ is given by a matrix $M \in \mathcal{G}(\mathbb{W}(\prod_{i\in I} O_{C_i})[\frac{1}{\pi}])$. Any φ-conjugation by a matrix $N \in \mathcal{G}(\mathbb{W}(\prod_{i\in I} O_{C_i})) = \prod_{i\in I}\mathcal{G}(\mathbb{W}(O_{C_i}))$ defines an isomorphic object in $(\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}})^{\dagger\mathrm{pre}}(S)$. This allows us to reduce to the case where the set I is a singleton and we must show that M is φ-conjugate by some $N \in \mathcal{G}(\mathbb{W}(O_C))$ to a matrix $M' \in \mathcal{G}(\mathbb{W}(C^+)[\frac{1}{\pi}])$. Let $k = O_C/C^{\circ\circ}$ and $k^+ = C^+/C^{\circ\circ}$ be the residue rings. As $\mathbb{W}(C^+)[\frac{1}{\pi}]$ is the preimage of $\mathbb{W}(k^+)[\frac{1}{\pi}]$ under $\mathbb{W}(O_C)[\frac{1}{\pi}] \to \mathbb{W}(k)[\frac{1}{\pi}]$, it suffices to prove the above claim after replacing O_C and C^+ by k and k^+, respectively. Indeed, if $\overline{M'} \in \mathcal{G}(\mathbb{W}(k^+)[\frac{1}{\pi}])$ can be conjugated to $\overline{M} \in \mathcal{G}(\mathbb{W}(k)[\frac{1}{\pi}])$ through a matrix $\overline{N} \in \mathcal{G}(\mathbb{W}(k))$, then any lift of $\overline{N}$ to $N \in \mathcal{G}(\mathbb{W}(O_C))$ would give the desired conjugation.

The claim is now that for every $\overline{M} \in \mathcal{G}(\mathbb{W}(k)[\frac{1}{\pi}])$ there is some $\overline{N} \in \mathcal{G}(\mathbb{W}(k)[\frac{1}{\pi}])$ such that

$$\overline{N}^{-1}\overline{M}\varphi(\overline{N}) \in \mathcal{G}(\mathbb{W}(k^+)[\frac{1}{\pi}])\,.$$

Note that k is an algebraically closed field, so that by the classification of isocrystals there are $A \in \mathcal{G}(\mathbb{W}(k)[\frac{1}{\pi}])$ and $T \in \mathcal{G}(\mathbb{W}(\overline{\mathbb{F}}_q)[\frac{1}{\pi}])$ such that $A^{-1}T\varphi(A) = \overline{M}$. By ind-properness of the Witt vector affine Grassmannian, we have $A = B \cdot C$ for some $B \in \mathcal{G}(\mathbb{W}(k^+)[\frac{1}{\pi}])$ and $C \in \mathcal{G}(\mathbb{W}(k))$. Letting $\overline{N} = C^{-1}$ we get

$$B^{-1}T\varphi(B) = \overline{N}^{-1}\overline{M}\varphi(\overline{N}).$$

Now $B^{-1}T\varphi(B) \in \mathcal{G}(\mathbb{W}(k^+)[\frac{1}{\pi}])$ as we wanted to show. □

Remark 7.13. This result can be regarded as a version of Fargues' theorem [Far18, Theorem 1.12] in families. Recall that Fargues' theorem states that the category of shtukas over (C, O_C) is equivalent to the category of BKF-modules of $\mathbb{W}(O_C)$. Although this statement is not true for general families, the theorem above shows that the statement is true v-locally. Indeed, $\mathrm{Sht}_{\mathcal{G}}(R, R^+)$ parametrizes $\mathcal{G}$-shtukas over $\mathrm{Spa}(R, R^+)$ while $(\mathrm{Sht}_{\mathcal{G}}^{\mathrm{sch}})^{\dagger}$ is the sheafification of the functor attaching to (R, R^+) the category of BKF-modules with $\mathcal{G}$-structure over $\mathbb{W}(R^{\circ})$.

7.2. The schematic comparison.

Theorem 7.14. *Let G be a reductive group over E and $\mathcal{G}$ a parahoric O_E-model of G.*

(1) The natural map $\mathfrak{B}(G) \xrightarrow{\cong} (\mathrm{Bun}_G)^{\mathrm{red}}$ is an isomorphism of scheme-theoretic v-sheaves valued in groupoids.

(2) The natural map $\mathrm{Sht}_{\mathcal{G}}^{\mathrm{sch}} \xrightarrow{\cong} (\mathrm{Sht}_{\mathcal{G}})^{\mathrm{red}}$ is an isomorphism of scheme-theoretic v-sheaves valued in groupoids.

Proof. Let $X \in \mathrm{PSch}^{\mathrm{aff}}$. For the first claim we write:

$$\begin{aligned}\mathfrak{B}(G)(X) &= \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_G, \mathfrak{B}(X))\\ &\cong \mathrm{Fun}^{\otimes}_{\mathrm{ex}}(\mathrm{Rep}_G, \mathrm{Bun}_{\mathrm{FF}}(X^{\diamond}))\\ &= \mathrm{Bun}_G(X^{\diamond})\\ &= (\mathrm{Bun}_G)^{\mathrm{red}}(X).\end{aligned}$$

Here, the second isomorphism is Proposition 4.7.

For the second claim, since $(\mathrm{Sht}_{\mathcal{G}})^{\mathrm{red}}$ and $\mathrm{Sht}_{\mathcal{G}}^{\mathrm{sch}}$ are v-sheaves (the latter by Proposition 6.3), it suffices to prove that $\mathrm{Sht}_{\mathcal{G}}^{\mathrm{sch}}(X) \xrightarrow{\cong} \mathrm{Sht}_{\mathcal{G}}(X^{\diamond})$ when $X = \mathrm{Spec}\, A$ is a comb. In this case, $\mathrm{Sht}_{\mathcal{G}}^{\mathrm{sch}}(X)$ is equivalent to the category where the objects are elements $M_{\mathcal{E}} \in \mathcal{G}(\mathbb{W}(A)[\frac{1}{\pi}])$, and morphisms between $M_{\mathcal{E}_1}$ and $M_{\mathcal{E}_2}$ are elements $N \in \mathcal{G}(\mathbb{W}(A))$ with $N^{-1}M_{\mathcal{E}_1}\varphi(N) = M_{\mathcal{E}_2}$. On the other hand, by Theorem 7.12.(5), an isomorphism between $M_{\mathcal{E}_1}$ and $M_{\mathcal{E}_2}$ in $\mathrm{Sht}_{\mathcal{G}}(X^{\diamond})$ corresponds to a functorial choice of elements $N_R \in \mathcal{G}(\mathbb{W}(R^{\circ}))$ with $N_R^{-1}M_{\mathcal{E}_1}\varphi(N_R) = M_{\mathcal{E}_2}$ ranging over maps $\mathrm{Spa}(R, R^+) \to X^{\diamond}$, with $\mathrm{Spa}(R, R^+)$ a product of points. Recall that the functor

$$(R, R^+) \mapsto R^{\circ}$$

is represented by the closed subsheaf $(\mathbb{A}^1)^{\dagger} \subseteq \mathrm{Spd}(\mathbb{F}_q[T], \mathbb{F}_q) = (\mathbb{A}^1_{\mathbb{F}_q})^{\Diamond}$. In particular, $H^0(X^{\diamond}, \mathcal{O}^{\circ}) \subseteq \{X^{\diamond} \to \mathbb{A}^1_{\mathbb{F}_q}\}$. By [Gle24, Theorem 2.32], we can conclude that $H^0(X^{\diamond}, \mathcal{O}^{\circ}) = A$. Since $H^0(X^{\diamond}, \mathcal{O}^{\circ}) = A$, such a collection of N_R uniquely comes from an element $N \in \mathcal{G}(\mathbb{W}(A))$. This shows that $\mathrm{Sht}_{\mathcal{G}}^{\mathrm{sch}}(X) \to \mathrm{Sht}_{\mathcal{G}}(X^{\diamond})$ is fully-faithful.

To prove essential surjectivity, fix $\mathcal{E} \in \mathrm{Sht}_{\mathcal{G}}(X^{\diamond})$. This induces elements $\mathcal{E}_{\mathrm{Bun}} \in \mathrm{Bun}_G(X^{\diamond})$ and $\mathcal{E}_{\mathfrak{B}} \in \mathfrak{B}(G)(X)$ unique up to isomorphism. Objects in $\mathrm{Sht}_{\mathcal{G}}^{\mathrm{sch}}(X)$ lifting $\mathcal{E}_{\mathfrak{B}}$ correspond to sections of $\mathrm{Sht}_{\mathcal{G}}^{\mathrm{sch}} \times_{\mathfrak{B}(G)} X \to X$, whereas objects in $\mathrm{Sht}_{\mathcal{G}}(X^{\diamond})$ lifting $\mathcal{E}_{\mathrm{Bun}}$ correspond to sections $\mathrm{Sht}_{\mathcal{G}} \times_{\mathrm{Bun}_G} X^{\diamond} \to X^{\diamond}$. The result then follows from Lemma 7.15 below. □

Lemma 7.15. *Let $X \in \mathrm{PSch}^{\mathrm{aff}}$ be a comb and $X \to \mathfrak{B}(G)$ be a map, then the natural map*

$$\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}} \times_{\mathfrak{B}(G)} X \xrightarrow{\simeq} (\mathrm{Sht}_{\mathcal{G}} \times_{\mathrm{Bun}_G} X^{\diamond})^{\mathrm{red}}$$

induces an isomorphism.

Proof. The argument given in [Gle22, Proposition 2.30] works in this generality. To orient the reader, we give a summary of the proof of [Gle22, Proposition 2.30] in the more general case we propose here. Since $X = \operatorname{Spec} A$ is a comb, every map $X \to \mathfrak{B}(G)$ is given by a matrix $M \in \mathcal{G}(\mathbb{W}(A)[\frac{1}{\pi}])$, cf. [Iva23, Theorem 6.1]. The functor

$$\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}} \times_{\mathfrak{B}(G)} X : (\mathrm{PSch}^{\mathrm{aff}})^{\mathrm{op}}_{/X} \to \mathrm{Sets}$$

parametrizes on $T = \operatorname{Spec} R$ triples $(\mathcal{E}^{\mathrm{sch}}, \Phi^{\mathrm{sch}}, \rho^{\mathrm{sch}})$, where $\mathcal{E}^{\mathrm{sch}}$ is a $\mathcal{G}$-torsor over $\operatorname{Spec} \mathbb{W}(R)$, $\Phi : \varphi^*\mathcal{E}^{\mathrm{sch}} \to \mathcal{E}^{\mathrm{sch}}$ is an isomorphism defined over $\operatorname{Spec} \mathbb{W}(R)[\frac{1}{\pi}]$ and $\rho : \mathcal{E}^{\mathrm{sch}} \to \mathcal{G}$ is an isomorphism defined over $\operatorname{Spec} \mathbb{W}(R)[\frac{1}{\pi}]$. Moreover, these data have to fit in the following commutative diagram

$$\begin{tikzcd} \varphi^*\mathcal{E}^{\mathrm{sch}} \arrow[r, "\varphi^*\rho^{\mathrm{sch}}"] \arrow[d, "\Phi^{\mathrm{sch}}"] & \mathcal{G} \arrow[d, "M_T"] \\ \mathcal{E}^{\mathrm{sch}} \arrow[r, "\rho^{\mathrm{sch}}"] & \mathcal{G}. \end{tikzcd}$$

We notice at this point that the data of Φ are completely determined by ρ, and for this reason $\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}} \times_{\mathfrak{B}(G)} X \simeq \mathrm{Gr}_X$, where the latter is the Witt vector Grassmannian which is an ind-scheme.

On the other hand, the functor

$$\mathrm{Sht}_{\mathcal{G}} \times_{\mathrm{Bun}_G} X^{\diamond} : (\mathrm{Perf}^{\mathrm{aff}})^{\mathrm{op}}_{/X^{\diamond}} \to \mathrm{Sets}$$

parametrizes on $S = \mathrm{Spa}(R, R^+)$ triples $(\mathcal{E}, \Phi, \rho)$, where $\mathcal{E}$ is a $\mathcal{G}$-torsor over $\mathcal{Y}_S$, the map $\Phi : \varphi^*\mathcal{E} \to \mathcal{E}$ is an isomorphism defined over $\mathcal{Y}_{(0,\infty),S}$ such that Φ is meromorphic along $V(\pi) \subseteq \mathcal{Y}_S$, the map $\rho : \mathcal{E} \to \mathcal{G}$ is an isomorphism defined over $\mathcal{Y}^{R^+}_{(0,\infty)}$ (not necessarily meromorphic), and the data fit in the commutative diagram

$$\begin{tikzcd} \varphi^*\mathcal{E} \arrow[r, "\varphi^*\rho"] \arrow[d, "\Phi"] & \mathcal{G} \arrow[d, "M_T"] \\ \mathcal{E} \arrow[r, "\rho"] & \mathcal{G}. \end{tikzcd}$$

Note that in this case, although the data of ρ determine Φ (since $\Phi = \rho^{-1} \circ M_T \circ \varphi^*\rho$), it is no longer true in this context that every choice of ρ defines a Φ which is meromorphic.

Now,

$$(\mathrm{Sht}_{\mathcal{G}} \times_{\mathrm{Bun}_G} X^{\diamond})^{\mathrm{red}} : (\mathrm{PSch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Sets}$$

has the formula

$$(\mathrm{Sht}_{\mathcal{G}} \times_{\mathrm{Bun}_G} X^{\diamond})^{\mathrm{red}}(T) = \mathrm{Sht}_{\mathcal{G}} \times_{\mathrm{Bun}_G} X^{\diamond}(T^{\diamond})$$

which is the same as giving a compatible system of maps

$$\mathrm{Spa}(R, R^+) \to \mathrm{Sht}_{\mathcal{G}} \times_{\mathrm{Bun}_G} X^{\diamond}$$

as $\mathrm{Spa}(R, R^+)$ ranges over over $\mathrm{Perf}^{\mathrm{aff}}_{/T^{\diamond}}$.

Given $(\mathcal{E}^{\mathrm{sch}}, \Phi^{\mathrm{sch}}, \rho^{\mathrm{sch}}) \in \mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}} \times_{\mathfrak{B}(G)} X(T)$ one can construct a compatible systems of maps by pulling back all data along $\mathcal{Y}_{\mathrm{Spa}(R,R^+)} \to \operatorname{Spa} \mathcal{Y}_T$, and what [Gle22, Proposition 2.30] shows is that every system of compatible data arises uniquely in this way. The proof of [Gle22, Proposition 2.30] treats the case in which $X = \operatorname{Spec} \bar{\mathbb{F}}_p$, but this hypothesis is not essential to the argument, the only thing that is really used is that the map $X \to \mathfrak{B}(G)$ has trivial underlying $\mathcal{G}$-torsor.

The proof of [Gle22, Proposition 2.30] goes as follows. One first shows that $\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}} \times_{\mathfrak{B}(G)} X \to (\mathrm{Sht}_{\mathcal{G}} \times_{\mathrm{Bun}_{\mathcal{G}}} X^\diamond)^{\mathrm{red}}$ is injective, or that the triple $(\mathcal{E}^{\mathrm{sch}}, \Phi^{\mathrm{sch}}, \rho^{\mathrm{sch}}) \in \mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}} \times_{\mathfrak{B}(G)} X(T)$ is determined by the compatible system of data that it induces. To prove surjectivity of $\mathrm{Sht}^{\mathrm{sch}}_{\mathcal{G}} \times_{\mathfrak{B}(G)} X \to (\mathrm{Sht}_{\mathcal{G}} \times_{\mathrm{Bun}_{\mathcal{G}}} X^\diamond)^{\mathrm{red}}$ the harder part of the argument is showing that given a compatible system of data

$$(\mathcal{E}_R, \Phi_R, \rho_R) \in (\mathrm{Sht}_{\mathcal{G}} \times_{\mathrm{Bun}_{\mathcal{G}}} X^\diamond)(R, R^+)$$

with $\mathrm{Spa}(R, R^+)$ ranging over $\mathrm{Perf}^{\mathrm{aff}}_{/T^\diamond}$, each of the ρ_R defined over $Y^{R^+}_{(0,\infty)}$ is meromorphic along $V(\pi) \subseteq Y^{R^+}_{[0,\infty)}$. Once one knows that each ρ_R is meromorphic, it is not hard to show it comes from a unique map

$$\rho_T : \mathcal{E}^{\mathrm{sch}} \to \mathcal{G}$$

defined over Y_T. □

7.3. **The topological comparison.** Recall from §6.2 that $B(G)$ is a partially ordered set. We equip $B(G)$ with the topology induced by the partial order, i.e. $[b] \in \overline{\{[b']\}}$ if and only if $[b] \leq [b']$ in the partial order. Recall that by results of Rapoport–Richartz [RR96] and He [He16, Theorem 2.12],

$$|\mathfrak{B}(G)| \cong B(G)\,. \tag{7.1}$$

Alternatively, by the results of Fargues–Scholze [FS24, Theorem I.4.1.(iii)] and of Viehmann [Vie23, Theorem 1.1], we also have

$$|\mathrm{Bun}_G|^{\mathrm{op}} \cong B(G)\,. \tag{7.2}$$

Here $|\mathrm{Bun}_G|^{\mathrm{op}}$ is the topological space where a subset is open in $|\mathrm{Bun}_G|^{\mathrm{op}}$ if and only if it is closed in $|\mathrm{Bun}_G|$. Combining these two sets of references, we obtain that

$$|\mathfrak{B}(G)| \cong |\mathrm{Bun}_G|^{\mathrm{op}}\,. \tag{7.3}$$

In this section we give a direct and new proof of the identity (7.3). As a consequence, we prove that the identities (7.1) and (7.2) are equivalent statements.

Let us clarify. The statements of the form

$$B(G) \cong |\mathfrak{B}(G)| \quad \text{ and } \quad B(G) \cong |\mathrm{Bun}_G|^{\mathrm{op}} \tag{7.4}$$

are naturally broken into two complementary statements. The first statement is Grothendieck's specialization theorem (as generalized in [RR96]), which says that the Newton polygon of a family of isocrystals over a scheme decreases as the family degenerates along a closed subscheme. The analogous statement for Bun_G is Kedlaya–Liu's semi-continuity theorem [SW20, Theorem 22.2.1] (as generalized in [FS24, Theorem I.4.1.(iii)]). The second statement is Grothendieck's conjecture [RR96, §3.13], which asks if every specialization of the combinatorially defined partial order in $B(G)$ can be realized by a geometric family. This is what [He16, Theorem 2.12] and [Vie23, Theorem 1.1] show in the $\mathfrak{B}(G)$ case and Bun_G case, respectively.

Our proof shows that if we assume that Grothendieck's specialization theorem (in its $\mathfrak{B}(G)$ and Bun_G form) already holds, Grothendieck's conjectures in both setups are equivalent.

Let us set some notation.

Definition 7.16. Let $b_1, b_2 \in B(G)$.

(1) We say that $b_1 \preceq_{\mathfrak{B}(G)} b_2$ if $b_1 \in \overline{\{b_2\}}$ in $\mathfrak{B}(G)$.
(2) We say that $b_1 \preceq_{\mathrm{Isoc}_G} b_2$ if $b_1 \in \overline{\{b_2\}}$ in Isoc_G.
(3) We say that $b_1 \preceq_{\mathrm{Bun}_G^{\mathrm{op}}} b_2$ if $b_2 \in \overline{\{b_1\}}$ in Bun_G.

Moreover, we write $b_1 \preceq_{\mathfrak{B}(G)} b_2$, $b_1 \preceq_{\mathrm{Isoc}_G} b_2$ or $b_1 \preceq_{\mathrm{Bun}_G^{\mathrm{op}}} b_2$ whenever b_2 covers b_1 in the respective partial order.

The following lemma says that the information of the topological spaces in question is completely determined by the partial order that the closure relations define. In particular, it will suffice to compare the corresponding partial orders.

Lemma 7.17. *Let $U \subseteq B(G)$. For $b \in B(G)$ we let $U_{\leq b} := U \cap B(G)_{\leq b}$.*

(1) U is closed in $\mathfrak{B}(G) \iff U_{\leq b}$ is closed in $\mathfrak{B}(G)$ for all $b \in B(G)$.
(2) U is closed in $\mathrm{Isoc}_G \iff U_{\leq b}$ is closed in Isoc_G for all $b \in B(G)$.
(3) U is open in $\mathrm{Bun}_G \iff U_{\leq b}$ is open in Bun_G for all $b \in B(G)$.
(4) $|\mathrm{Bun}_G|^{\mathrm{op}}$ is a topological space.
(5) The topologies on $\mathfrak{B}(G)$, Isoc_G and Bun_G are determined by the partial order they induced as in Definition 7.16. More precisely a set $U \subset \mathfrak{B}(G)$ is closed if and only if for all $b \in U$ the sets $\mathfrak{B}(G)_{\leq b} = \{b' \in \mathfrak{B}(G) \mid b' \leq b\}$ is a subset of U (analogously for Isoc_G and Bun_G).

Proof. We prove the first claim, the second and third claim being analogous. The forward implication is evident since $\mathfrak{B}(G)_{\leq b} \subseteq \mathfrak{B}(G)$ is a closed immersion. Let $f : \operatorname{Spec} R \to \mathfrak{B}(G)$ be any map. As $\operatorname{Spec} R$ is quasi-compact, there are a finite number of elements $b_i^f \in B(G)$ such that f factors through $\bigcup_{i=1}^n \mathfrak{B}(G)_{\leq b_i^f}$. By assumption $U \cap \bigcup_{i=1}^n \mathfrak{B}(G)_{\leq b_i^f}$ is closed in $\mathfrak{B}(G)$. Since f factors through the set above, base change of f along U defines a closed immersion.

The fourth claim follows from the third. Indeed, the only part that needs justification is that an arbitrary union of open subsets in $|\mathrm{Bun}_G|^{\mathrm{op}}$ is open. This is equivalent to the preservation of open subsets of $|\mathrm{Bun}_G|$ under arbitrary intersections, but arbitrary intersections can be expressed as finite intersections when we restrict them to $\mathrm{Bun}_G^{\leq b}$.

The last claim follows from the first three. Indeed, $\mathfrak{B}(G)$, Isoc_G and Bun_G have the strong topology along the inclusion maps from $\coprod_{b \in B(G)} \mathfrak{B}(G)_{\leq b}$, $\coprod_{b \in B(G)} \mathrm{Isoc}_G^{\leq b}$ and $\coprod_{b \in B(G)} \mathrm{Bun}_G^{\leq b}$. Moreover, since these latter ones are finite topological spaces, they are determined by their closure relations. $\square$

The following implications are reformulations of Grothendieck's specialization theorem in the respective setup.

(1) $b_1 \preceq_{\mathfrak{B}(G)} b_2 \implies b_1 \leq_{B(G)} b_2$
(2) $b_1 \preceq_{\mathrm{Isoc}_G} b_2 \implies b_1 \leq_{B(G)} b_2$
(3) $b_1 \preceq_{\mathrm{Bun}_G^{\mathrm{op}}} b_2 \implies b_1 \leq_{B(G)} b_2$

Theorem 7.18. *The partial orders $\preceq_{\mathfrak{B}(G)}$, $\preceq_{\mathrm{Isoc}_G}$, $\preceq_{\mathrm{Bun}_G^{\mathrm{op}}}$ agree. In particular, we have natural homeomorphisms*

$$|\mathfrak{B}(G)| \cong |\mathrm{Isoc}_G| \cong |\mathrm{Bun}_G|^{\mathrm{op}}.$$

Proof. For the rest of the proof we fix $b_1, b_2 \in B(G)$ with $b_1 \leq_{B(G)} b_2$. We first prove $|\mathfrak{B}(G)| \cong |\mathrm{Isoc}_G|$. Recall that $\Diamond$ preserves closed immersions, consequently:

$$b_1 \preceq_{\mathrm{Isoc}_G} b_2 \implies b_1 \preceq_{\mathfrak{B}(G)} b_2 .$$

Now, suppose that $b_1 \preceq_{\mathfrak{B}(G)} b_2$. We claim that there is a perfect rank 1 valuation ring V and a map $\operatorname{Spec} V \to \mathfrak{B}(G)$ such that the induced maps on $\operatorname{Spec} k_V$ (the residue field) and $\operatorname{Spec} K_V$ (the fraction field) factor through $\mathfrak{B}(G)_{b_1}$ and $\mathfrak{B}(G)_{b_2}$, respectively. Indeed, since we assumed $b_1 \preceq_{\mathfrak{B}(G)} b_2$, we may find a map $f : \operatorname{Spec} R \to \mathfrak{B}(G)$ with the property that for all $x \in \operatorname{Spec} R$ the induced map $\operatorname{Spec} k_x \to \mathfrak{B}(G)$ factors through either $\mathfrak{B}(G)_{b_1}$ or $\mathfrak{B}(G)_{b_2}$ and with the property that $\overline{f^{-1}(\mathfrak{B}(G)_{b_2})} \cap f^{-1}(\mathfrak{B}(G)_{b_1}) \neq \emptyset$. We may replace $\operatorname{Spec} R$ by a v-cover, so we may assume that $R = \prod_{i \in I} V_i$ is a product of valuation rings. Since the inclusion $\mathfrak{B}(G)_{\leq b_1} \to \mathfrak{B}(G)$ is perfectly finitely presented, there is $r \in R$ such that $\operatorname{Spec} R/(r)^{\mathrm{perf}} \subseteq$

$\operatorname{Spec} R$ is equal to $f^{-1}(\mathfrak{B}(G)_{b_1})$. We may write $R = R_1 \times R_2$, where $R_1 = \prod_{\{i\in I | r_i = 0\}} V_i$ and $R_2 = \prod_{\{i\in I | r_i \neq 0\}} V_i$, and replace R by R_2. Let K_{V_i} denote the fraction field of V_i. Now, $\operatorname{Spec} \prod_{i\in I} K_{V_i} \subseteq \operatorname{Spec} R$ is a pro-open subset lying in $f^{-1}(\mathfrak{B}(G)_{b_2})$. Since $f^{-1}(\mathfrak{B}(G)_{b_1})$ is non-empty, there is a connected component in $x \in \beta I$ with associated valuation ring V_x such that that the image of r in V_x, which we denote by r_x, is not identically 0, but is also not a unit. The largest prime ideal contained in $\langle r_x\rangle$ and the smallest prime ideal containing $\langle r_x\rangle$ define a rank 1 valuation ring with the desired properties.

The map $\operatorname{Spec} V \to \mathfrak{B}(G)$ induces a map $\operatorname{Spd}(V, V) \to \mathfrak{B}(G)^\diamond \to \operatorname{Isoc}_G$ such that the corresponding maps on $\operatorname{Spd}(k_V, k_V)$ and $\operatorname{Spd}(K_V, K_V)$ factor through $\operatorname{Isoc}_G^{b_1}$ and $\operatorname{Isoc}_G^{b_2}$, respectively. This implies that $\operatorname{Spd}(K_V, V) \to \operatorname{Isoc}_G$ factors through $\operatorname{Isoc}_G^{b_2}$, but $\operatorname{Spd}(K_V, V) \subseteq \operatorname{Spd}(V, V)$ is dense. This proves:

$$b_1 \preceq_{\mathfrak{B}(G)} b_2 \implies b_1 \preceq_{\operatorname{Isoc}_G} b_2 .$$

In the same fashion, the map $\operatorname{Spec} V \to \mathfrak{B}(G)$ induces a map $\operatorname{Spd}(V, V) \to \mathfrak{B}(G)^\diamond \to \operatorname{Bun}_G$ that if restricted to $\operatorname{Spd}(k_V, k_V)$ and $\operatorname{Spd}(K_V, K_V)$, factors through $\operatorname{Bun}_G^{b_1}$ and $\operatorname{Bun}_G^{b_2}$, respectively. Let $\pi \in V$ be a pseudo-uniformizer, let $\hat{V}_\pi$ be the π-adic completion of V and let $K = \hat{V}_\pi[\frac{1}{\pi}]$. We note that $\operatorname{Spa}(K, \hat{V}_\pi)$ is a perfectoid field and that $\operatorname{Spd}(\hat{V}_\pi, \hat{V}_\pi)$ has two points, one corresponding to $\operatorname{Spd}(K, \hat{V}_\pi)$ and one corresponding to $\operatorname{Spd}(k_V, k_V)$. By Theorem 3.15, the map $\operatorname{Spd}(\hat{V}_\pi, \hat{V}_\pi) \to \operatorname{Bun}_G$ corresponds to a $\otimes$-exact functor from Rep_G to the category of φ-equivariant objects in $\operatorname{Vect}(Y^K_{(0,\infty]})$. Using [PR24, Proposition 2.1.3] and [FS24, Theorem II.2.14], we conclude that the map $\operatorname{Spd}(\hat{V}_\pi, \hat{V}_\pi) \to \operatorname{Bun}_G$ factors through $\operatorname{Bun}_G^{b_1}$ as $\operatorname{Spd}(k_V, k_V) \to \operatorname{Bun}_G$ does. Moreover, $\operatorname{Spd}(\hat{V}_\pi, \hat{V}_\pi) \subseteq \operatorname{Spd}(V, V)$ is an open subsheaf whose v-sheaf-theoretic closure is $\operatorname{Spd}(V, V)$. This allows us to conclude:

$$b_1 \preceq_{\mathfrak{B}(G)} b_2 \implies b_1 \preceq_{\operatorname{Bun}_G^{\mathrm{op}}} b_2 .$$

Finally, suppose that $b_1 \preceq_{\operatorname{Bun}_G^{\mathrm{op}}} b_2$. Using these assumptions we may find a map $\operatorname{Spa}(R, R^+) \to \operatorname{Bun}_G$ with the property that for all $x \in \operatorname{Spa}(R, R^+)$, the induced map $\operatorname{Spa}(C_x, C_x^+) \to \operatorname{Bun}_G$ factors through either $\operatorname{Bun}_G^{b_1}$ or $\operatorname{Bun}_G^{b_2}$ and with the property that $\overline{f^{-1}(\operatorname{Bun}_G^{b_1})} \cap f^{-1}(\operatorname{Bun}_G^{b_2}) \neq \emptyset$. Replacing $\operatorname{Spa}(R, R^+)$ by a v-cover, we may assume that it is a product of points with $R^+ = \prod_{i\in I} C_i^+$. By shrinking $\operatorname{Spa}(R, R^+)$ and ignoring some factors if necessary, we may assume that the principal components of $\operatorname{Spa}(R, R^+)$ all factor through $\operatorname{Bun}_G^{b_1}$ without changing the condition that $\overline{f^{-1}(\operatorname{Bun}_G^{b_1})} \cap f^{-1}(\operatorname{Bun}_G^{b_2}) \neq \emptyset$. This forces at least one non-principal component to factor through $\operatorname{Bun}_G^{b_2}$. Moreover, we may assume $C_i^+ = O_{C_i}$ for all i so that $R^+ = R^\circ$. By Theorem 7.12, we may assume that our map $\operatorname{Spa}(R, R^+) \to \operatorname{Bun}_G$ is induced from a map $\operatorname{Spec} R^+ \to \mathfrak{B}(G)$. Let k_i denote the residue field of O_{C_i}. By assumption, the map $\operatorname{Spa}(C_i, O_{C_i}) \to \operatorname{Bun}_G$ factors through $\operatorname{Bun}_G^{b_1}$. In particular, $\operatorname{Spec} k_i \to \mathfrak{B}(G)$ factors through $\mathfrak{B}(G)_{b_1}$, which implies that $\operatorname{Spec} \prod_{i\in I} k_i \to \mathfrak{B}(G)$ also factors through $\mathfrak{B}(G)_{b_1}$. Indeed, it certainly factors through $\mathfrak{B}(G)_{\leq b_1}$, and the locus where it factors through $\mathfrak{B}(G)_{b'}$ with $b' < b_1$ is finitely presented and contains no principal component of $\operatorname{Spec} \prod_{i\in I} k_i$, which implies that it is empty. We see that the closed point of every connected component of $\operatorname{Spec} R^+$ factors through $\mathfrak{B}(G)_{b_1}$. Furthermore, there is at least one point $x \in \operatorname{Spec} R^+$ mapping to $\mathfrak{B}(G)_{b_2}$. The connected component containing x defines a valuation ring V_x and a map $\operatorname{Spec} V_x \to \mathfrak{B}(G)$ such that the closed point factors through $\mathfrak{B}(G)_{b_1}$ and at least one point of $\operatorname{Spec} V_x$ factors through $\mathfrak{B}(G)_{b_2}$. This allows us to conclude that

$$b_1 \preceq_{\operatorname{Bun}_G^{\mathrm{op}}} b_2 \implies b_1 \preceq_{\mathfrak{B}(G)} b_2 . \qquad \square$$

Recall that the groupoid of maps $X \to \mathfrak{B}(G)$ and the groupoid of maps $X^\diamond \to \operatorname{Bun}_G$ are equivalent. The following statement explains the relation between the stratifications that such data induces in X.

Proposition 7.19. *Let $X = \operatorname{Spec} A$ and let $X \to \mathfrak{B}(G)$ be a map. Let $Z_b \subseteq X$ denote the closed subscheme that factors through $\mathfrak{B}(G)_{\leq b}$ and let $U_b \subseteq X^\diamond$ denote the open subsheaf that factors through $\mathrm{Bun}_G^{\leq b}$. Then $U_b = \widehat{X}_{/Z_b}$ with notation as in* [Gle24, Definition 4.18] *(i.e. U_b is the formal neighborhood around Z_b).*

Proof. By definition U_b is an open subsheaf and since $Z_b \to X$ is a finitely presented closed immersion, $\widehat{X}_{/Z_b}$ is also an open subsheaf by [Gle24, Proposition 4.22] . It suffices to show that U_b and $\widehat{X}_{/Z_b}$ have the same geometric points and since both of these subsheaves are partially proper over $X^\diamond$, it even suffices to show that they have the same rank 1 geometric points. Every map of the form $\operatorname{Spa}(C, O_C) \to X^\diamond$ factors uniquely through a map $\operatorname{Spd} O_C \to X^\diamond$. We let $k = O_C/C^{\circ\circ}$ and note that $|\operatorname{Spd} O_C|$ consists of two points, where one corresponds to $\operatorname{Spa}(C, O_C)$ while the other corresponds to $\operatorname{Spd} k$. Let $b_k \in B(G)$ be the unique isomorphism class of the induced map $\operatorname{Spd} k \to X^\diamond \to \mathrm{Bun}_G$ We claim that $\operatorname{Spd} O_C \to \mathrm{Bun}_G$ factors through $\mathrm{Bun}_G^{b_k}$. Indeed, if $S = \operatorname{Spa}(C, O_C)$, by Theorem 3.15 the map $\operatorname{Spd} O_C \to \mathrm{Bun}_G$ corresponds to a $\otimes$-exact functor from Rep_G to the category of φ-equivariant objects in $\mathcal{V}(Y_{(0,\infty],S})$. The claim then follows from [SW20, Theorem 13.2.1, Theorem 13.4.1].

In particular, a rank 1 geometric point $\operatorname{Spa}(C, O_C) \to X^\diamond$ lies over U_b if and only if its induced $\operatorname{Spd} k \to X^\diamond$ point lies over $\mathrm{Bun}_G^{b_k}$ for $b_k \in B(G)_{\leq b}$. This is the same as saying that its image under the specialization map

$$\mathrm{sp}: |X^\diamond| \to |X|$$

lies on $|Z_b| \subseteq |X|$. This in turn is the definition of $\widehat{X}_{/Z_b}$. □

Corollary 7.20. *Let $X = \operatorname{Spec} A$ and let $X \to \mathfrak{B}(G)$ be a map. Let $U \subseteq B(G)$ the intersection of a finite closed subset with an open subset. Let $Z_U \subseteq X$ denote the locally closed subscheme that factors through $U \subseteq |\mathfrak{B}(G)|$. Let $U_b \subseteq X^\diamond$ denote the locally closed subsheaf that factors through $\mathrm{Bun}_G^{b \in U}$. Then U_b is the smallest subsheaf of $X^\diamond$ containing $\widehat{X^\diamond}_{/Z_b}$ and stable under vertical specialization.*

Proof. The same argument as in Proposition 7.19 will show that $\widehat{X^\diamond}_{/Z_b}$ and U_b agree on rank 1 points. Since we allow U to be more general, it is no longer true that $\widehat{X^\diamond}_{/Z_b}$ is partially proper over $X^\diamond$, while it is still true that U_b is. The description given above takes this into account. □

Appendix A. Sheaves of categories

Throughout the body of the text, we used the sheafification construction in $\mathrm{Cat}_{1,E}^{\otimes,\mathrm{ex}}$. This appendix has two purposes: 1) to justify why the ∞-category $\mathrm{Cat}_{1,E}^{\otimes,\mathrm{ex}}$ is presentable and compactly generated so that sheafification is well defined and 2) to collect some general categorical statements pertaining separated presheaves that were used in the above. We start with the second purpose. For the rest of the appendix we let $\mathcal{T} \in \{\mathrm{Perf}^{\mathrm{aff}}, \mathrm{PSch}^{\mathrm{aff}}\}$.

Recall that given a complete ∞-category $\mathcal{C}$ and a presheaf $\mathcal{F} \in \mathcal{P}(\mathcal{T}, \mathcal{C})$ we say that $\mathcal{F}$ is a sheaf if for every object $X \in \mathcal{T}$ and every covering sieve $\mathcal{U} \subseteq \mathcal{T}_{/X}$ the natural map

$$\mathcal{F}(X) \to \varprojlim_{U \in \mathcal{U}} \mathcal{F}(U)$$

is an equivalence. The following statement says that in our categories $\{\mathrm{Perf}^{\mathrm{aff}}, \mathrm{PSch}^{\mathrm{aff}}\}$, it suffices to verify that descent holds for specific covering sieves.

Lemma A.1. *A presheaf $\mathcal{F} \in \mathcal{P}(\mathcal{T}, \mathcal{C})$ is a sheaf if and only if for every map $[U \to X] \in \mathcal{T}$ the map*

$$\mathcal{F}(X) \to \mathrm{Desc.}(\mathcal{F}, U/X)$$

is an equivalence.

Here $\mathrm{Desc.}(\mathcal{F}, U/X)$ denotes the descent category defined in terms of the Čech resolution $U_\bullet \to X$, where U_i denotes the i-fold product $U_i := U \times_X \cdots \times_X U$.

Proof. This is classical. For a reference, see [HM24, Lemma A.4.8, A.4.6]. □

Lemma A.2. *Let $\mathcal{F}_1, \mathcal{F}_2 : \mathcal{T} \to \mathrm{Cat}_1$ be two sheaves of categories on $\mathcal{T}$ and suppose that $f : \mathcal{F}_1 \to \mathcal{F}_2$ is a morphism of sheaves. If f is fully-faithful and locally on $\mathcal{T}$ essentially surjective, then it is essentially surjective.*

Proof. Let $X \in \mathcal{T}$ and let $(X_i \to X)_i$ be a covering in $\mathcal{T}$, such that f_{X_i} is essentially surjective. For any $B \in \mathcal{F}_2(X)$, let $A_i \in F_1(X_i)$ be a preimage under f_{X_i} of $B|_{X_i}$. By the sheaf property of $\mathcal{F}_1$ and the fully-faithfulness of f_{X_i} and $f_{X_i \times_X X_j}$ it follows that the A_i glue to a unique object of $F(X)$ mapping to B. □

Definition A.3. Let C be a presentable category. Let $\mathcal{F} \in \mathcal{P}(\mathcal{T}, C)$ be a presheaf and let $\tilde{\mathcal{F}}$ denote its sheafification. We call $\mathcal{F}$ *separated* if the natural map $\mathcal{F} \to \mathcal{F} \times_{\tilde{\mathcal{F}}} \mathcal{F}$ in $\mathcal{P}(\mathcal{T}, C)$ is an equivalence.

Recall the $+$-construction (or $\dagger$-construction) involved in the explicit construction of sheafification [Lur09, 6.2.2.9-6.2.2.13]. The functor $\mathcal{F} \mapsto \mathcal{F}^+$ comes with a natural transformation $\eta_{(-)} : \mathrm{id} \to (-)^+$ which satisfies the property that for all $\mathcal{G} \in \mathcal{S}(\mathcal{T}, C)$ and all $\mathcal{F} \in \mathcal{P}(\mathcal{T}, C)$ the maps

$$\mathrm{Hom}_{\mathcal{P}(\mathcal{T},C)}(\mathcal{F}^+, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{P}(\mathcal{T},C)}(\mathcal{F}, \mathcal{G})$$

are equivalences [Lur09, Lemma 6.2.2.14]. In particular, if $\mathcal{F}^+$ is a sheaf the map $\eta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^+$ exhibits $\mathcal{F}^+$ as the sheafification of $\mathcal{F}$. We use the following criterion to recognize a separated presheaf.

Proposition A.4. *Let C be a presentable and compactly generated category and let $\mathcal{F} \in \mathcal{P}(\mathcal{T}, C)$. Given a covering map $[f : U \to X] \in \mathcal{T}$, we let $\mathcal{D}_{U/X} :=$ Desc.$(\mathcal{F}, U/X)$. The following are equivalent.*

(1) $\mathcal{F}$ is a separated presheaf.
(2) For all f as above, the diagonal $\Delta_{\mathcal{F},f} : \mathcal{F}(X) \to \mathcal{F}(X) \times_{\mathcal{D}_{U/X}} \mathcal{F}(X)$ is an equivalence.

Moreover, if any of these statements hold, then $\mathcal{F}^+$ is a sheaf.

Proof. We start by showing that (2) already implies that $\mathcal{F}^+$ is a sheaf. Given $A \in C^\omega$ we let $h_A \in \mathcal{P}(\mathcal{T}, \mathrm{Ani})$ denote the presheaf of mapping anima $h_A(X) := \mathrm{Hom}_C(A, \mathcal{F}(X))$. We observe that since $A \in C^\omega$, we have functorial equivalences $(h_A)^+ \simeq \mathrm{Hom}_C(A, \mathcal{F}^+)$. By the Yoneda embedding [Lur09, Proposition 5.1.3.1], the family of functors $\{\mathrm{Hom}(A, -) : C \to \mathrm{Ani}\}_{A \in C^\omega}$ is conservative. In particular, we may test the sheaf condition on $\mathcal{F}^+$ by showing that h_A^+ is a sheaf for all $A \in C^\omega$. The hypothesis show that for every map $[U \to X] \in \mathcal{T}$ the diagonal $h_A(X) \to h_A(X) \times_{\mathrm{Desc.}(h_A, U/X)} h_A(X)$ is an equivalence. This suffices to conclude that h_A^+ is a sheaf (see [AS20, Proposition 3.4.22]), and consequently $\mathcal{F}^+$ also is.

Let us show that the two statements are equivalent. We observe that since C is compactly generated, the sheafification of h_A agrees with $\mathrm{Hom}_C(A, \tilde{\mathcal{F}})$. Since both assertions, the first being $\mathcal{F}$ being separated, and the second being the condition that $\Delta_{\mathcal{F},f} : \mathcal{F}(X) \to \mathcal{F}(X) \times_{\mathcal{D}_{U/X}} \mathcal{F}(X)$ is an equivalence, can be verified on the mapping anima presheaves $\{h_A\}_{A \in C^\omega}$, we may and do reduce to the case $C = \mathrm{Ani}$ and $\mathcal{F} = h_A$. In one direction, arguing as above, if the diagonal $h_A(X) \to h_A(X) \times_{\mathrm{Desc.}(h_A, U/X)} h_A(X)$ is an equivalence, then the sheafification of h_A is h_A^+, one can use the concrete expression of $h_A^+(X)$ and that in Ani filtered colimits commute with finite limits to show that $h_A \to h_A \times_{\tilde{h}_A} h_A$ is also an equivalence.

For the converse, fix a cover $[U \to X] \in \mathcal{T}$ and consider the diagram

$$\begin{tikzcd}
\mathrm{Desc.}(\mathcal{F}, U/X) \arrow[r, "s'"] & \mathrm{Desc.}(\widetilde{\mathcal{F}}, U/X) \\
\mathcal{F}(X) \arrow[u, "\alpha"] \arrow[r, "s"'] & \widetilde{\mathcal{F}}(X). \arrow[u, "\tilde{\alpha}"']
\end{tikzcd}$$

We note that for a map $f : c \to d$ in an ∞-category, the diagonal $\Delta_f : c \to c \times_d c$ is an equivalence if and only if f is (-1)-truncated if and only if f is a monomorphism. In general, n-truncated morphisms have the left cancellation property and we know that s and s' are monomorphisms by assumption and $\widetilde{\alpha}$ is even an equivalence. This implies that α is also (-1)-truncated.

□

Proposition A.5. *Suppose that $\mathcal{C}$ is presentable and compactly generated category. Then the subcategory of $\mathcal{P}(\mathcal{T}, \mathcal{C})$ spanned by separated presheaves $\mathcal{F}$ is stable under finite limits.*

Proof. Arguing as in the proof of Proposition A.4 we may reduce to the case $\mathcal{C} = \mathrm{Ani}$. This case follows from the fact that sheafification is exact. □

Lemma A.6. *Suppose that $\mathcal{F} \in \mathcal{P}(\mathcal{T}, \mathrm{Cat}_{1,E}^{\otimes,\mathrm{ex}})$. The following statements are equivalent.*

(1) $\mathcal{F}$ is separated as an object in $\mathcal{P}(\mathcal{T}, \mathrm{Cat}_1)$.
(2) $\mathcal{F}$ is separated as an object in $\mathcal{P}(\mathcal{T}, \mathrm{Cat}_{1,E}^{\otimes})$.
(3) $\mathcal{F}$ is separated as an object in $\mathcal{P}(\mathcal{T}, \mathrm{Cat}_{1,E}^{\otimes,\mathrm{ex}})$.

Proof. The claim is easily reduced to showing that a map $[A \to B] \in \mathrm{Cat}_{1,E}^{\otimes,\mathrm{ex}}$ satisfies that the diagonal $\Delta_f : A \to A \times_B A$ is an equivalence if and only if $\Delta_{\mathcal{F}(f)} : \mathcal{F}(A) \to \mathcal{F}(A) \times_{\mathcal{F}(B)} \mathcal{F}(A)$ is when we denote $\mathcal{F} : \mathrm{Cat}_{1,E}^{\otimes,\mathrm{ex}} \to \mathrm{Cat}_1$ (resp. $\mathcal{F} : \mathrm{Cat}_{1,E}^{\otimes,\mathrm{ex}} \to \mathrm{Cat}_{1,E}^{\otimes}$) the forgetful functor.

As the forgetful functor commutes with limits, we have $\Delta_{\mathcal{F}(f)} \simeq \mathcal{F}(\Delta_f)$. Moreover, the inverse of $\mathcal{F}(\Delta_f)$ is necessarily given by $\mathcal{F}(\pi_1)$, where $\pi_1 : A \times_B A \to A$ is the projection onto the first factor. This shows that $\mathcal{F}(\Delta_f)^{-1}$ is automatically E-linear and exact as we wanted to show. □

The previous considerations allow us to give a complicated but concrete description of the categories that one obtains from applying sheafification to a separated presheaf with values in $\mathrm{Cat}_{1,E}^{\otimes,\mathrm{ex}}$.

Lemma A.7. *Suppose that $\mathcal{F} \in \mathcal{P}(\mathcal{T}, \mathrm{Cat}_{1,E}^{\otimes,\mathrm{ex}})$ is a separated presheaf. Let $\widetilde{\mathcal{F}} \in \mathcal{S}(\mathcal{T}, \mathrm{Cat}_{1,E}^{\otimes,\mathrm{ex}})$ be its sheafification. For all $S \in \mathcal{T}$, the category $\widetilde{\mathcal{F}}(X) \in \mathrm{Cat}_{1,E}^{\otimes,\mathrm{ex}}$ can be described as follows: Objects $V \in \widetilde{\mathcal{F}}(S)$ can be described as tuples (S', V', α), where $[S' \to S] \in \mathcal{T}$ is a v-cover and $V' \in$ $\mathrm{Desc.}(\mathcal{F}, S'/S)$. If we have a map $S'' \xrightarrow{f} S' \to S$ with $S'' \to S$ a v-cover, then (S', V', α) is isomorphic to $(S'', f^*V', f^*\alpha)$. Given two objects, (S', V_1', α_1) and (S', V_2', α_2) the set of morphisms between them agrees with the set of morphisms in $\mathrm{Desc.}(\mathcal{F}, S'/S)$. A sequence $\Sigma := [(S', V_1', \alpha_1) \to (S', V_2', \alpha_2) \to (S', V_3', \alpha_3)]$ is exact if and only if there is $S'' \xrightarrow{f} S' \to S$ with $S'' \to S$ a v-cover such that the sequence $f^*\Sigma := [f^*V_1' \to f^*V_2' \to f^*V_3']$ is exact in $\mathcal{F}(S'')$.*

Lemma A.8. *Let $\mathcal{F} \in \mathcal{P}(\mathcal{T}, \mathrm{Cat}_{1,E}^{\otimes,\mathrm{ex}})$ be a separated presheaf with sheafification $\widetilde{\mathcal{F}}$. Then the sheafification of $\mathrm{Fun}_E^{\otimes,\mathrm{ex}}(\mathrm{Rep}_G, \mathcal{F})$ is $\mathrm{Fun}_E^{\otimes,\mathrm{ex}}(\mathrm{Rep}_G, \widetilde{\mathcal{F}})$.*

Proof. The category Rep_G is generated under tensor products by finitely many objects, say $\{\mathcal{O}_i\}_{i=1}^n$ with I finite. We claim that for any $S \in \mathrm{Perf}^{\mathrm{aff}}$ and $F \in \mathrm{Fun}^{\otimes,\mathrm{ex}}(\mathrm{Rep}_G, \widetilde{\mathcal{F}})(S)$, there exists a v-cover $S' \to S$ such that F factors through a (exact monoidal) functor $\mathrm{Rep}_G \to \mathrm{Desc.}(\mathcal{F}, S'/S)$. Indeed, according to Lemma A.7 each of the tensor generators $\mathcal{O}_i$ maps to an object of the form (S_i, V_i, α_i) and we can take $S' = S_1 \times_S \cdots \times S_n$. Furthermore, since for every other $S'' \to S' \to S$ the map

$$\mathrm{Desc.}(\mathcal{F}, S'/S) \to \mathrm{Desc.}(\mathcal{F}, S''/S)$$

is fully-faithful, all morphisms between the tensor powers of the $\mathcal{O}_i$ must lie in $\mathrm{Desc.}(\mathcal{F}, S'/S)$.

With other words, we get the first equality in

$$\begin{aligned}\mathrm{Fun}^{\otimes,\mathrm{ex}}(\mathrm{Rep}_G, \widetilde{\mathcal{F}})(S) &= \mathrm{colim}_{S' \to S} \mathrm{Fun}^{\otimes,\mathrm{ex}}(\mathrm{Rep}_G, \mathrm{Desc.}(\mathcal{F}, S'/S)) \\ &= \mathrm{colim}_{S' \to S} \mathrm{Desc.}(\mathrm{Fun}^{\otimes,\mathrm{ex}}(\mathrm{Rep}_G, \mathcal{F}), S'/S),\end{aligned}$$

where the second equality holds as Desc.$(-, S'/S)$ is a limit and Fun commutes with limits. The last expression is the value of the sheafification of $\mathrm{Fun}^{\otimes,\mathrm{ex}}(\mathrm{Rep}_G, \mathcal{F})$ on S, so we are done. □

Remark A.9. One can be much more general in the formulation of Lemma A.8. Indeed, given appropriate cut-off cardinals $\kappa < \lambda$, the categories $\mathrm{Perf}^{\mathrm{aff}}_{\lambda}$ and $\mathrm{PSch}^{\mathrm{aff}}_{\lambda}$ have enough κ-cofiltered limits. Consequently, $\mathrm{Fun}^{\otimes,\mathrm{ex}}(\mathrm{Rep}_G, -)$ commutes with sheafification for general $\mathcal{F}$ as long as Rep_G is κ-compact. There is always a κ for which this holds. To avoid discussing further technicalities, we have settled with the formulation and proof given above.

A.1. **The category** $\mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E}$**.** We verify that $\mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E}$ is presentable and compactly generated in several steps. We will also define $\mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E}$ in steps, and at each step we will show that the relevant category is presentable and compactly generated. Throughout, we will use the following statement.

Theorem A.10 ([RS22, Theorem 1.1])**.** *Let $\mathcal{C}$ be a presentable ∞-category and let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory which is closed under limits and κ-filtered colimits for some regular cardinal κ. Then, $\mathcal{D}$ is presentable.*

We can combine this with the following useful statement.

Proposition A.11. *Let $\mathcal{C}$ be a compactly generated presentable ∞-category, let $\mathcal{D}$ be a presentable ∞-category, let $\mathcal{F}\colon \mathcal{D} \to \mathcal{C}$ be a conservative functor which commutes with limits and filtered colimits. Then, $\mathcal{D}$ is compactly generated.*

Proof. Using the adjoint functor theorem and the hypothesis, we conclude that $\mathcal{F}\colon \mathcal{D} \to \mathcal{C}$ admits a left adjoint $L\colon \mathcal{C} \to \mathcal{D}$. Let $\{K_i\}_{i\in I} \subseteq \mathcal{C}$ be a family of compact generators for $\mathcal{C}$. We claim that $\{L(K_i)\}_{i\in I}$ is a family of compact generators for $\mathcal{D}$. Indeed, they are compact since $\mathcal{F}$ preserves filtered colimits. To show they generate $\mathcal{D}$, consider the full subcategory $\mathcal{D}_0 \subseteq \mathcal{D}$ generated under finite colimits by objects of the form $L(K_i)$. By [Lur09, Corollary 5.3.4.15], $\mathcal{D}_0$ consists of compact objects. We obtain a fully-faithful functor $\iota\colon \mathrm{Ind}\,\mathcal{D}_0 \to \mathcal{D}$ that commutes with filtered colimits, and hence commutes with all colimits by [Lur09, Proposition 5.5.1.9]. By the adjoint functor theorem [Lur09, Corollary 5.5.2.9], we get a right adjoint $G\colon \mathcal{D} \to \mathrm{Ind}\,\mathcal{D}_0$. To show that ι is an equivalence, it suffices to show that G is conservative. We observe that L factors along ι and consequently $\mathcal{F}$ factors along G. Since $\mathcal{F}$ was assumed to be conservative, G is also conservative. □

Corollary A.12. *Let $\mathcal{C}$ be a compactly generated presentable ∞-category and let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory which is closed under limits and filtered colimits. Then, $\mathcal{D}$ is presentable and compactly generated.*

Proof. By Theorem A.10 above, $\mathcal{D}$ is presentable. The claim now follows from Proposition A.11. □

A.1.1. *Additive categories.*

Definition A.13. An ∞-category $\mathcal{C}$ is *semiadditive* if it is pointed, admits finite products and coproducts, and for every finite collection $\{C_s \in \mathcal{C}\}_{s\in S}$ the norm map

$$\coprod_{s\in S} C_s \to \prod_{t\in S} C_t$$

described in [Lur17, Example 6.1.6.11] is an equivalence. An ∞-category $\mathcal{C}$ is additive if it is semiadditive and its homotopy category is additive in the classical sense.

Since being semiadditive (or additive) is a property of an ∞-category, we could consider the full subcategory of Cat_∞ spanned by these objects. Nevertheless, this ∞-category will contain too many functors. Indeed, one is only interested in those functors that respect the semiadditive structure. As it turns out it suffices to look at functors that preserve finite coproducts.

Definition A.14. We denote by SemiAdd the ∞-category of semiadditive ∞-categories (which is defined through e.g. [Har20, Corollary 5.4]) and by Add its full subcategory of additive categories.

Proposition A.15. *The ∞-categories* Add *and* SemiAdd *are compactly generated.*

Proof. Since Add is stable under limits and filtered colimits in SemiAdd, we are reduced to showing that the ∞-category of semiadditive ∞-categories SemiAdd together with additive functors is compactly generated by Corollary A.12. By [Har20, Corollary 5.4], we are reduced to showing that $\mathrm{Cat}(K_0)$ (the category of categories with finite coproducts and functors preserving them) is compactly generated. By [Lur17, Lemma 4.8.4.2], $\mathrm{Cat}(K_0)$ is presentable. We can apply Proposition A.11 to the forgetful functor $\mathrm{Cat}(K_0) \to \mathrm{Cat}_\infty$ to conclude that $\mathrm{Cat}(K_0)$ is compactly generated. Indeed, by [Lur09, Corollary 5.3.6.10] the forgetful functor admits a left adjoint. Further, given a filtered diagram $I \to \mathrm{Cat}(K_0)$, we wish to show that $C = \varinjlim_{i\in I} C_i \in \mathrm{Cat}_\infty$ stays in $\mathrm{Cat}(K_0)$ and that for every $i \in I$ the projection map $C_i \to C$ is functor in $\mathrm{Cat}(K_0)$. This can be done as in the proof of [Lur09, Proposition 5.5.7.11]. More precisely, given a finite collection $\{o_s\}_{s\in S} \subseteq C$, we may lift it to a finite collection $\{o_{s,i}\}_{s\in S}$ for some $i \in I$. The image of the coproduct of the $\{o_{s,i}\}_{s\in S}$ is a coproduct in C. $\square$

Within the category of additive ∞-categories, we have the subcategory $\mathrm{Add}_{\leq 1} \subseteq \mathrm{Add}$ of classical additive categories. It can be realized as the full subcategory of 1-truncated objects. Since being 1-truncated is stable under limits and filtered colimits, it follows that $\mathrm{Add}_{\leq 1}$ is compactly generated and that the inclusion $\mathrm{Add}_{\leq 1} \subseteq \mathrm{Add}$ has a left adjoint.

A.1.2. *Symmetric monoidal additive categories.* As justified in [Har20, Proposition 5.6] and [Lur17, Proposition 4.8.2.7], there is a symmetric monoidal structure $\mathrm{SemiAdd}^\otimes \to N(\mathrm{Fin}_*)$ on the ∞-category SemiAdd. This product captures the following phenomena: Given categories $C_1, C_2, \mathcal{D} \in \mathrm{SemiAdd}$, then $\mathrm{Fun}(C_1 \otimes C_2, \mathcal{D})$ captures functors $G \in \mathrm{Fun}(C_1 \times C_2, \mathcal{D})$ that are additive in both variables (i.e. $G(c_1 \oplus c_1', c_2) \simeq G(c_1, c_2) \oplus G(c_1', c_2)$ and $G(c_1, c_2 \oplus c_2) \simeq G(c_1, c_2) \oplus G(c_1, c_2'))$. One can verify that both $\mathrm{Add}_{\leq 1}$ and Add are preserved under $\otimes$. In particular, we obtain symmetric monoidal structures $\mathrm{Add}^\otimes_{\leq 1}$ and $\mathrm{Add}^\otimes$ on $\mathrm{Add}_{\leq 1}$ and Add, respectively.

We know from [Lur17, Lemma 4.8.4.2] that $\otimes\colon\ \mathrm{Add}_{\leq 1} \times \mathrm{Add}_{\leq 1} \to \mathrm{Add}_{\leq 1}$ preserves colimits in each variable. It follows from [Lur17, Corollary 3.2.3.5] that the categories of commutative algebra objects $\mathrm{CAlg}(\mathrm{Add}^\otimes)$ and $\mathrm{CAlg}(\mathrm{Add}^\otimes_{\leq 1})$ are again presentable. Moreover, by [HM24, Lemma B.2.4], the forgetful functor $\mathrm{CAlg}(\mathrm{Add}^\otimes_{\leq 1}) \to \mathrm{Add}_{\leq 1}$ is conservative and commutes with limits and filtered colimits. Using Proposition A.11, we conclude that $\mathrm{CAlg}(\mathrm{Add}^\otimes_{\leq 1})$ is compactly generated.

Objects in $\mathrm{CAlg}(\mathrm{Add}^\otimes_{\leq 1})$ capture the data of an additive category C together with a symmetric monoidal operation $\otimes\colon C \times C \to C$ that is additive on both variables, and a $\otimes$-unit $\mathbb{1} \in C$ satisfying the usual further compatibilities. Functors in $\mathrm{CAlg}(\mathrm{Add}^\otimes_{\leq 1})$ capture symmetric monoidal additive functors between such categories. We will now isolate in $\mathrm{CAlg}(\mathrm{Add}^\otimes_{\leq 1})$ those categories that are rigid.

Definition A.16. Given $(C, \otimes, \mathbb{1})$ a symmetric monoidal ∞-category, we say that an object $o \in C$ is *dualizable* if there exists a dual $o^\vee$ together with an evaluation map $\mathrm{ev}_o\colon\ o \times o^\vee \to \mathbb{1}$, a coevaluation map $\mathrm{coev}_o\colon\ \mathbb{1} \to o \otimes o^\vee$ and commutative diagrams

$$\begin{tikzcd} o \arrow[r, "\mathrm{coev}_o\otimes \mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & o \otimes o^\vee \otimes o \arrow[d, "\mathrm{id}\otimes \mathrm{ev}_o"] \\ & o \end{tikzcd} \qquad \begin{tikzcd} o^\vee \arrow[r, "\mathrm{id}\otimes \mathrm{coev}_o"] \arrow[dr, "\mathrm{id}"'] & o^\vee \otimes o \otimes o^\vee \arrow[d, "\mathrm{ev}_o\otimes \mathrm{id}"] \\ & o^\vee . \end{tikzcd}$$

We say that $(C, \otimes, \mathbb{1})$ is *rigid* if every object of C is dualizable.

One can easily verify that symmetric monoidal functors preserve dualizable objects, in particular we can form the full subcategory of $\mathrm{Cat}_1^\otimes \subseteq \mathrm{CAlg}(\mathrm{Add}^\otimes_{\leq 1})$ spanned by those symmetric monoidal categories that are rigid. It follows from Corollary A.12 that $\mathrm{Cat}_1^\otimes$ is presentable and compactly generated.

Given a ring R, for example $R \in \{E, O_E\}$, we let $\mathrm{Proj}(R) \in \mathrm{Cat}_1^{\otimes}$ denote the rigid symmetric monoidal additive category of finite projective R-modules. We let $\mathrm{Cat}_{1,R}^{\otimes}$ denote the coslice category of $\mathrm{Cat}_1^{\otimes}$ under $\mathrm{Proj}(R)$. Since the forgetful functor $\mathrm{Cat}_{1,R}^{\otimes} \to \mathrm{CAlg}(\mathrm{Add}_{\leq 1}^{\otimes})$ commutes with limits and filtered colimits, applying Theorem A.10 and Proposition A.11, we get that $\mathrm{Cat}_{1,R}^{\otimes}$ is also compactly generated. Given a map of rings $R \to R'$ and a category $C \in \mathrm{Cat}_{1,R}^{\otimes}$, we let $C \otimes_R R'$ denote the short hand for $C \otimes_{\mathrm{Proj}(R)} \mathrm{Proj}(R')$.

A.1.3. *Exact categories.* In the above, we have justified that $\mathrm{Cat}_{1,O_E}^{\otimes}$ and $\mathrm{Cat}_{1,E}^{\otimes}$ are presentable and compactly generated. In what follows, we will justify that $\mathrm{Cat}_{1,O_E}^{\otimes,\mathrm{ex}}$ and $\mathrm{Cat}_{1,E}^{\otimes,\mathrm{ex}}$ are also presentable and compactly generated.

Recall that Quillen's definition of an exact category can be rephrased in ∞-categorical language in terms of Waldhausen and coWaldhausen categories, cf. [Bar15, Example 3.3].

Definition A.17. [Bar15, Definition 3.1] An exact ∞-category consists of a triple $(C, C_\dagger, C^\dagger)$ such that:

(1) The underlying ∞-category C is additive.
(2) The pair $(C, C_\dagger)$ is a Waldhausen ∞-category.
(3) The pair $(C, C^\dagger)$ is a coWaldhausen ∞-category.
(4) A square in C is an ambigressive pullback if and only if it is an ambigressive pushout.

Here we use the following terminology:

- $C_\dagger \subseteq C$ is a wide subcategory and a morphism of $C_\dagger$ is called ingressive or a cofibration.
- $C^\dagger \subseteq C$ is a wide subcategory and a morphism of $C^\dagger$ is called egressive or a fibration.
- A pullback square
$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, two heads] \\ X' \arrow[r, tail] & Y' \end{tikzcd}$$
is said to be ambigressive if $X' \rightarrowtail Y'$ is ingressive and $Y \twoheadrightarrow Y'$ is egressive. Dually, a pushout square
$$\begin{tikzcd} X \arrow[r, tail] \arrow[d, two heads] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd}$$
is said to be ambigressive if $X \rightarrowtail Y$ is ingressive and $X \twoheadrightarrow X'$ is egressive.
- An exact sequence is fiber/cofiber sequence
$$\begin{tikzcd} X' \arrow[r, tail] \arrow[d, two heads] & X \arrow[d, two heads] \\ 0 \arrow[r, tail] & X''. \end{tikzcd}$$

Proposition A.18. *The ∞-category* Exact *of exact ∞-categories is compactly generated, and the forgetful functor* Exact $\to$ Add *commutes with limits and filtered colimits.*

Proof. By definition, we have a fully faithful embedding $\mathrm{Exact} \hookrightarrow \mathrm{Wald}_\infty \times_{\mathrm{Cat}_\infty} \mathrm{coWald}_\infty$. The categories Wald_∞ and coWald_∞ are compactly generated by [Bar16, Proposition 4.8]. Moreover, by [Bar16, Propositions 4.4, 4.5] and the adjoint functor theorem, it follows that the maps $\mathrm{Wald}_\infty \to \mathrm{Cat}_\infty$ and $\mathrm{CoWald}_\infty \to \mathrm{Cat}_\infty$ are maps in Pr^R. By [Lur09, Proposition 5.5.7.6], it follows that $\mathrm{Wald}_\infty \times_{\mathrm{Cat}_\infty} \mathrm{coWald}_\infty$ is also compactly generated. Hence by Corollary A.12, it suffices to show the above embedding preserves limits and filtered colimits.

Let $p\colon I \to \mathrm{Exact}$ be a diagram with the compositions $\pi_1 \circ p\colon I \to \mathrm{Wald}_\infty$ and $\pi_2 \circ p\colon I \to \mathrm{coWald}_\infty$, where π_1(resp. π_2)$\colon \mathrm{Exact} \to \mathrm{Wald}_\infty$(resp. coWald_∞) are the forgetful functors. Let $(\mathcal{C}, \mathcal{C}_\dagger) = \varprojlim \pi_1 \circ p$ and $(\mathcal{C}, \mathcal{C}^\dagger) = \varprojlim \pi_2 \circ p$. We want to show that $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ is an exact category. By [Bar16, Proposition 4.5], it follows that $(\mathcal{C}, \mathcal{C}_\dagger) = (\varprojlim_{i\in I} \mathcal{C}_i, \varprojlim_{i\in I} \mathcal{C}_{\dagger,i})$ and dually $(\mathcal{C}, \mathcal{C}^\dagger) = (\varprojlim_{i\in I} \mathcal{C}_i, \varprojlim_{i\in I} \mathcal{C}_i^\dagger)$. Since additive categories are stable under limits, it suffices to show that a square is an ambigressive pullback if and only if it is an ambigressive pushout. We know this is true for the categories $(\mathcal{C}_i, \mathcal{C}_{\dagger,i}, \mathcal{C}_i^\dagger)$. We have an isomorphism of morphism ∞-categories

$$\mathrm{Mor}_{\varprojlim_{i\in I} \mathcal{C}_i}(\varprojlim_{i\in I} x_i, \varprojlim_{i\in I} y_i) \simeq \varprojlim_{i\in I} \mathrm{Mor}_{\mathcal{C}_i}(x_i, y_i)\,.$$

But then, a square is a pullback/pushout square if and only if its projections to the categories $\mathcal{C}_i$ are pullback/pushout squares. To show this, we use [Lur09, Proposition 4.4.2.6] to reduce to the case of products and fiber products. For products, this follows from [Lur09, Corollary 5.1.2.3] and for fiber products this follows from [Lur09, Proposition 5.4.5.5]. Similarly, for a filtered ∞-category I with a functor $p'\colon I \to \mathrm{Exact}$ with compositions $\pi_1 \circ p'\colon I \to \mathrm{Wald}_\infty$ and $\pi_2 \circ p'\colon I \to \mathrm{coWald}_\infty$, we let $(\mathcal{C}, \mathcal{C}_\dagger) = \varinjlim \pi_1 \circ p'$ and $(\mathcal{C}, \mathcal{C}^\dagger) = \varinjlim \pi_2 \circ p'$. We want to show that $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ is an exact category. Again using [Bar16, Proposition 4.5], it follows that $(\mathcal{C}, \mathcal{C}_\dagger) = (\varinjlim_{i\in I} \mathcal{C}_i, \varinjlim_{i\in I} \mathcal{C}_{\dagger,i})$ and dually $(\mathcal{C}, \mathcal{C}^\dagger) = (\varinjlim_{i\in I} \mathcal{C}_i, \varinjlim_{i\in I} \mathcal{C}_i^\dagger)$. Now let $x_{i_0} \in \mathcal{C}_{i_0}$ and $y_{i_1} \in \mathcal{C}_{i_1}$ with their images $x, y \in \mathcal{C}$. By [Roz12], there is an isomorphism of morphism ∞-categories

$$\mathrm{Mor}_{\mathcal{C}}(x, y) \simeq \varinjlim_{j \in I_{(i_0, i_1)/}} \mathrm{Mor}_{\mathcal{C}_j}(x_j, y_j)\,. \tag{A.1}$$

We consider an ambigressive pushout

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d, two heads] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd}$$

in $\mathcal{C}$. We can pick a finite level $i_0 \in I$, and lifts $[x_{i_0} \to x'_{i_0}], [x_{i_0} \to y_{i_0}] \in \mathcal{C}_{i_0}$ of the maps $[X \to Y], [X \to X']$ such that $[x_{i_0} \to y_{i_0}]$ is ingressive and $x_{i_0} \to x'_{i_0}$ is egressive. By [Bar16, Definition 2.7], pushouts of ingressive morphisms exist. We let y'_{i_0} denote the pushout. Since pushout diagrams of ingressive morphisms remain ingressive under any functor in Wald_∞, we can show that the image of y'_{i_0} under $\mathcal{C}_{i_0} \to \mathcal{C}$ is equivalent to Y' by using Equation (A.1). Since $\mathcal{C}_{i_0}$ is exact, we have an ambigressive pushout diagram which is also an ambigressive pullback diagram

$$\begin{tikzcd} x_{i_0} \arrow[r, tail] \arrow[d, two heads] & y_{i_0} \arrow[d, two heads] \\ x'_{i_0} \arrow[r, tail] & y'_{i_0}\,. \end{tikzcd}$$

Using Equation (A.1) and that functors in coWald_∞ preserve egressive pullbacks, one can show that the image of the square above in $\mathcal{C}$ remains an egressive pullback. This argument shows that if a square in $\mathcal{C}$ is an ambigressive pushout, then it is an ambigressive pullback. One can use the dual argument to show the converse. $\square$

We can finally define $\mathrm{Cat}_{1,E}^{\otimes,\mathrm{ex}}$ by considering the Cartesian square in $\widehat{\mathrm{Cat}}_\infty$

$$\begin{tikzcd} \mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E} \arrow[r] \arrow[d] & \mathrm{Exact} \arrow[d, "\mathcal{F}_2"] \\ \mathrm{Cat}^{\otimes}_{1,E} \arrow[r, "\mathcal{F}_1"] & \mathrm{Add}\ . \end{tikzcd}$$

Proposition A.19. *The category* $\mathrm{Cat}^{\otimes,\mathrm{ex}}_{1,E}$ *is presentable and compactly generated.*

Proof. Both forgetful functors $\mathcal{F}_1$ and $\mathcal{F}_2$ are maps in Pr^R as in [Lur09, Definition 5.5.3.1], since they commute with small limits and filtered colimits. By [Lur09, Theorem 5.5.3.18], Pr^R admits small limits and they can be computed in $\widehat{\mathrm{Cat}}_\infty$. Moreover, by [Lur09, Proposition 5.5.7.6] for $\kappa = \omega$, fiber products in Pr^R preserve the property of being compactly generated. $\square$

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